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INTRODUCTION

TO

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INTRODUCTION

TO

ANALYTICAL MECHANICS

BY

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PREFACE.

Οὐχ ὡς θέλομεν, ἀλλ' ὡς δυνάμεθα.

The present volume is intended as a brief introduction to mechanics for junior and senior students in colleges and universities. It is based to a large extent on Ziwet's Theoretical Mechanics; but the applications to engineering are omitted, and the analytical treatment has been broadened. No knowledge of differential equations is presupposed, the treatment of the occurring equations being fully explained. It is believed that the book can readily be covered in a three-hour course extending throughout a year. For a shorter course, requiring half this time, the following selection may be made: Chapters 1, 2, 3 (omitting Arts. 81–95), 4 (omitting Arts. 114–150), 5 to 12 (omitting Arts. 244–268), 13 and 14 (omitting Arts. 340–355).

While more prominence has been given to the analytical side of the subject, the more intuitive geometrical ideas are generally made to precede the analysis. In doing this the idea of the vector is freely used; but it has seemed best to avoid the special methods and notations of vector analysis. This has been done with reluctance; the time has certainly come for introducing these methods in the very elements of mechanics. But this must be left to another opportunity.

That many important subjects had to be omitted is another restriction arising from the nature and purpose of this volume. While the selection of topics has been considered most carefully it can hardly be expected to meet everybody's approval. The aim has been not only to select material useful to the beginning student of mathematics and physical science, but

at the same time to give the reader a general view of the science of mechanics as a whole, a broad enough foundation for further study.

References to other works have been used sparingly. It seemed hardly necessary to refer to such standard works as those of Thomson and Tait, Routh, Schell, Appell, Kirchhoff, etc., which are found in any good college library. But it did seem desirable to refer in a few cases to works where fuller information can be found on subjects somewhat out of the range of the ordinary text-book on mechanics. The fourth volume of the *Encyklopädie der mathematischen Wissenschaften*, especially the articles by P. Stäckel, should be consulted by the more advanced student.

Alexander Ziwet,
Peter Field.

University of Michigan,
February, 1912.

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INTRODUCTION.

1. The science of **mechanics** can be regarded as an extension of geometry obtained by adjoining the ideas of **time** and **mass** to the idea of **space** which is fundamental in geometry. We are thus led to the study of motion and of forces as the subject-matter of mechanics.

2. By adjoining the idea of time alone we obtain a preliminary branch of mechanics, known as **kinematics.** It develops the ideas of velocity and acceleration of geometrical configurations without using the notion of mass.

3. The introduction of mass leads to numerous new ideas such as momentum, force, energy. Owing to the importance of forces in physics the mechanics of bodies possessing mass is often called **dynamics.** It may be divided into statics and kinetics.

Statics is the science of equilibrium; it considers the conditions under which the action of forces produces no change of motion. Thus, if force be regarded as a fundamental concept, statics is independent of the idea of time.

Kinetics treats in the most general way the changes of motion produced by forces.

PART I: KINEMATICS.

CHAPTER I.

RECTILINEAR MOTION OF A POINT.

1. Velocity and acceleration in rectilinear motion.

4. Consider the motion of a point P along a fixed straight line (Fig. 1). If we take on this line an origin O and a definite positive sense, say toward the right from O, the *position* of the point P on the line at any time t can be assigned by its *co-ordinate*, or *abscissa*, $OP = s$, which may be

Fig. 1.

any real number. As P moves along the line its abscissa s varies with the time; to every value of t (at least within a certain interval of time) corresponds a certain value of s; in other words, s is a function of t. We assume that s is a *continuous* function of t; this implies that while P may move arbitrarily, back and forth, along the line, it does not make any jumps, suddenly disappearing at one point and reappearing at another; the path of P is connected.

5. *The time-rate of change of the abscissa of* P, *i. e.* the t-derivative of s, *is called the* **velocity** of the point P; it is usually denoted by the letter v:

$$v = \frac{ds}{dt}.$$

As the idea of velocity is fundamental in mechanics it may be well to explain somewhat more in detail the genesis of this idea, the more so as the process is typical and recurs frequently.

Let the point P move along the line, or any segment of the line, always in the same sense and so that equal distances are always described in equal times. Such a motion is called *uniform*, and the quotient s/t of any distance $OP = s$ described, divided by the corresponding time t, is called the *velocity of the uniform motion:*

$$v = \frac{s}{t}.$$

Suppose next that the point P does not move uniformly. The same quotient, $v = s/t$, of any distance described, divided by the time used in describing it, is now called the *average*, or *mean*, *velocity for that distance or time.* This mean velocity varies in general according to the distance or time selected; it does not characterize the motion as a whole. We can, however, attach a definite meaning to the expression *velocity at a given point or instant* if we define it as follows.

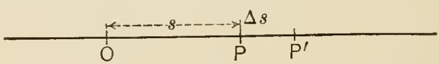

Fig. 2.

Let $s = OP$ (Fig. 2) be the abscissa of the moving point at the time t, $s + \Delta s = OP'$ its abscissa at the time $t + \Delta t$, so that the distance Δs is described by P in the time Δt; and let Δt be taken so small that P moves always in the same sense as it describes the distance Δs. Then $\Delta s/\Delta t$ is the *mean* velocity for the distance Δs or time Δt. The limit approached by this quotient as Δt approaches zero.

$$v = \lim_{\Delta t = 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

is called the *velocity at the point P*, or at the time t.

It is assumed that such a limit exists, *i. e.*, that s is a differentiable function of t.

The definition of velocity as the time-rate of change of the co-ordinate s applies even in the case of uniform motion; for in this case we have as stated above

$$s = vt,$$

where v is a constant, *i. e.* independent of t; hence

$$\frac{ds}{dt} = v.$$

In non-uniform, or variable, motion the velocity v varies from point to point and from time to time; it can be regarded as a function of the distance s or of the time t.

It should be observed that in this whole discussion of velocity it is not essential that the path be rectilinear, this assumption being made only for the sake of simplicity. The discussion applies without change when the point P describes a curve; the co-ordinate s then means the arc of the curve measured along the curve from some origin O on the curve, a definite sense of progression along the curve being taken as positive.

6. Velocity being defined as the quotient of distance by time in uniform motion, and as the limit of such a quotient in any motion, the **unit of velocity** is the unit of length divided by the unit of time. Thus we speak of a velocity of so many centimeters per second (cm./sec.), or feet per minute (ft./min.), or miles per hour (M./h.), etc.

Denoting the units of time, length, and velocity by T, L, V, this is expressed symbolically by writing

$$V = \frac{L}{T} = LT^{-1}$$

and saying that the **dimensions** of velocity (V) are 1 in length (L) and -1 in time (T).

The reader is supposed to be familiar with the C.G.S. (centimeter-gram-second) and F.P.S. (foot-pound-second) systems of measurement. It will suffice to mention that the *second* is the $\frac{1}{86400}$ part of the mean solar day which is the average, for one year, of the time between two successive passages of the sun across the meridian; and that the *foot* is $\frac{1}{3}$ of a yard, the American yard being defined (by act of

Congress, 1866) as $\frac{3600}{3937}$ of a meter. We have therefore the *exact* relations

$$1 \text{ cm.} = 0.3937 \text{ in.}, \quad \frac{\text{ft.}}{\text{cm.}} = \frac{1200}{39.37},$$

which give approximately:

$$1\text{m.} = 3.2808 \text{ ft.}, \ 1 \text{ ft.} = 30.48 \text{ cm.}, \ 1 \text{ in.} = 2.54 \text{ cm.}$$

7. Exercises.

(1) Compare the following velocities by reducing all to ft./sec.: (*a*) man walking 4 M./h.; (*b*) horse trotting a mile in 2 min. 10 sec.; (*c*) train running 40 M./h.; (*d*) ship making 15 knots, a knot being a sea-mile (= 6080 ft.) per hour; (*e*) sound in dry air at 0° C. 331.3 m./sec.; (*f*) sun moving in space 25 km./sec.; (*g*) light 3×10^{10} cm./sec.

(2) Two men starting (in opposite sense) from the same point walk around a block forming a rectangle of sides a, b; if their constant velocities are v_1, v_2, when and where will they meet?

(3) The mean distance of the sun being $92\frac{1}{3}$ million miles, find the velocity of light if it takes light 16 min. 42 sec. to cross the earth's orbit: (*a*) in miles per second, (*b*) in kilometers per hour.

(4) Two trains, one 250, the other 420 ft. long, pass each other on parallel tracks in opposite sense, with equal velocities. A passenger in the shorter train observes that it takes the longer train just 6 sec. to pass him. What is the velocity?

(5) What is the distance from A to B if a man walking 5 M./h. can cover it in 10 min. less than one walking 3 M./h.?

(6) What is the answer to the preceding problem if both men start from A at the same time and, when the one has reached B, the other is $7\frac{1}{2}$ miles behind him?

(7) Two ships start from the same port, the second an hour later than the first. The velocity of the first is 16 knots, that of the second 14 knots. How many miles are they apart 3 hours after the first ship started, the angle between their paths being 60°?

8. The definition of velocity

$$v = \frac{ds}{dt}$$

enables us to find the velocity when the co-ordinate s is

given as a function of t. Conversely, when v is given as a function of t or of s (or of both s and t), the integration of the same equation gives a relation between s and t which determines the position of the moving point at any time.

Thus, if v is given as a function of t, we find by integrating the relation $ds = vdt$:

$$s - s_0 = \int_{t_0}^{t} vdt,$$

where s_0 is the position of the moving point at the time t_0, the so-called **initial position.**

If v is given as a function of s, we find by integrating the relation $dt = ds/v$:

$$t - t_0 = \int_{s_0}^{s} \frac{ds}{v}.$$

9. Exercises.

(1) Find the velocity when: (*a*) $s = at + b$, (*b*) $s = at^2 + bt + c$, (*c*) $s = a\sqrt{t}$, (*d*) $s = a \cos kt$, (*e*) $s = ae^{-t}$, (*f*) $s = \frac{1}{2}a(e^t + e^{-t})$, (*g*) $s = \frac{1}{6}a(2t^3 + 3t^2 + 1)$, (*h*) $s = a(t^2 - 1)^2$, (*i*) $s = at^2(t - 1)$, (*j*) $s = a(t^5 - 20t^2 - 1)$, (*k*) $s = at/(1 - t^2)$. Taking a as a positive constant, discuss the motion by determining when s and v have maxima or minima. The nature of the motion will be best understood by sketching in each case the curve that represents s as a function of t, and then imagining this curve projected on the axis of s. Analytically, the sign of the velocity determines the sense of the motion; *i. e.* when v is > 0, s increases; when $v < 0$, s decreases; when $v = 0$ and $dv/dt \neq 0$, s has an extreme value and the sense of the motion changes.

(2) Find the distance s in terms of t when: (*a*) $v = v_0 + gt$, (*b*) $v = a(t^2 - 4)$, (*c*) $v = a \sec^2 t$, (*d*) $v = -\dfrac{s_0 t}{\sqrt{1 - t^2}}$, (*e*) $v = ae^{\alpha t + \beta}$; with $s = s_0$ for $t = 0$.

(3) Find t as a function of s and s as a function of t when: (*a*) $v = \sqrt{2gs}$, with $s = s_0$ for $t = 0$; (*b*) $v = \sqrt{a^2 - s^2}$, with $s = 0$ when $t = 0$; (*c*) $v = \sqrt{a^2 + s^2}$, with $s = 0$ for $t = 0$.

10. *In rectilinear motion, the time-rate of change of the velocity is called the* **acceleration**; denoting it by the letter j,

we have

$$j = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

We are led to the idea of acceleration by a process of reasoning strictly analogous to that followed in defining velocity (Art. 5). Among non-uniform motions, the most simple kind is that in which the velocity always increases (or always decreases) by equal amounts in equal times; it is called *uniformly accelerated motion*. In this kind of motion, the quotient obtained by dividing the increase (or decrease) of the velocity in any time by this time is called the *acceleration of the uniformly accelerated motion*.

If the motion is *not* uniformly accelerated the same quotient is called the *average*, or *mean, acceleration for that time*. Thus, if the velocity is v at the time t when the moving point has the position P, and reaches the value $v + \Delta v$ at the subsequent time $t + \Delta t$, when the point is at P', the mean acceleration in the time Δt (or distance $PP' = \Delta s$) is $\Delta v/\Delta t$. The limit of this quotient, as Δt approaches zero, *i. e.*

$$j = \lim_{\Delta t = 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt},$$

is called the *acceleration at the time* t (or at the distance s).

It follows that *in uniformly accelerated motion the acceleration is constant;* and conversely, when the acceleration is constant, the motion is uniformly accelerated.

11. A rectilinear motion is called *accelerated* whether the velocity be increasing or decreasing. But sometimes the term *acceleration* is used in a more restricted sense, as opposed to *retardation*. The motion is then called accelerated or retarded according as the *absolute value* of the velocity is increasing or diminishing. This gives the criterion

$$\frac{d(v^2)}{dt} \gtrless 0, \quad i.\ e.\ v\frac{dv}{dt} \gtrless 0.$$

Thus *the motion is accelerated* (in the narrower sense) *or retarded according as vdv/dt is* > 0 *or* < 0; if $dv/dt = 0$ while

$d^2v/dt^2 \neq 0$, the motion changes from being accelerated to being retarded, or vice versa.

Acceleration being defined, for rectilinear motion, as the quotient of velocity by time or as the limit of such a quotient, the *unit of acceleration* J is the unit of velocity divided by the unit of time. With the notation of Art. 6, this is expressed symbolically by writing

$$J = \frac{V}{T} = \frac{L}{T^2} = LT^{-2};$$

hence the *dimensions* of acceleration are 1 in length and -2 in time. Thus we speak of an acceleration of so many centimeters per second per second (cm./sec.^{2}).

12. Exercises.

(1) A point moving with constant acceleration gains at the rate of 30 M./h. in every minute. Express its acceleration in ft./sec.^{2}.

(2) At a place where the acceleration of gravity is $g = 9.810$ m./sec.^{2}, what is the value of g in ft./sec.^{2}?

(3) A railroad train, 10 min. after starting, attains a velocity of 45 M./h.; what is its average acceleration during these 10 min.?

(4) How does the acceleration of gravity which is about 32.2 ft./sec.^{2} compare with that of the train in Ex. (3)?

(5) Find the acceleration for the motions in Art. 9, Ex. (1); apply the rule of Art. 11 to determine where each motion is accelerated or retarded.

(6) Discuss in the same way the acceleration of the motions in Art. 9, Ex. (2) and (3).

13. A rectilinear motion is fully characterized if its acceleration is given as a function of t, s, v, provided that the **initial conditions** are also given, viz. the position and velocity of the moving point at any instant.

For we then have

$$\frac{d^2s}{dt^2} = j(t, s, v), \tag{1}$$

where $v = ds/dt$, while $j(t, s, v)$ is a given function. The solution of this differential equation, which is called the **equation of motion**, with the initial conditions $s = s_0$, $v = v_0$ for $t = t_0$, gives s as a function of t.

The solution of such a differential equation may be difficult; nor can any general rule of procedure be given. We here confine ourselves to very simple cases, especially those where the acceleration j is either a constant or a function of s alone, these cases being most important.

2. Examples of rectilinear motion.

14. Uniformly Accelerated Motion. As in this case the acceleration j is constant (see Art. 10), the equation of motion (1)

$$\frac{d^2s}{dt^2} = j, \quad \text{or} \quad \frac{dv}{dt} = j,$$

can readily be integrated:

$$v = jt + C.$$

To determine the constant of integration C, we must know the value of the velocity at some particular instant. Thus, if $v = v_0$ when $t = 0$, we find $v_0 = C$; hence, substituting this value for C,

$$v - v_0 = jt. \tag{2}$$

This equation gives the velocity at any time t. Substituting ds/dt for v and integrating, we find $s = v_0t + \frac{1}{2}jt^2 + C'$, where the constant of integration, C', must again be determined from given "initial conditions." Thus, if we know that $s = s_0$ when $t = 0$, we find $s_0 = C'$; hence

$$s - s_0 = v_0t + \tfrac{1}{2}jt^2. \tag{3}$$

This equation gives the space or distance passed over in terms of the time.

Eliminating j between (2) and (3), we obtain the relation

$$s - s_0 = \tfrac{1}{2}(v_0 + v)t,$$

which shows that *in uniformly accelerated motion the space can be found as if it were described uniformly with the mean velocity* $\tfrac{1}{2}(v_0 + v)$.

15. To obtain the velocity in terms of the space, we have only to eliminate t between (2) and (3); we find

$$\tfrac{1}{2}(v^2 - v_0^2) = j(s - s_0). \tag{4}$$

This relation can also be derived by eliminating dt between the differential equations $v = ds/dt$, $dv/dt = j$, which gives $vdv = jds$, and integrating. The same equation (4) is also obtained directly from the fundamental equation of motion $d^2s/dt^2 = j$ by a process very frequently used in mechanics, viz. by multiplying both members of the equation by ds/dt. This makes the left-hand member the exact derivative of $\tfrac{1}{2}(ds/dt)^2$ or $\tfrac{1}{2}v^2$, and the integration can therefore be performed.

16. The three equations (2), (3), (4) contain the complete solution of the problem of uniformly accelerated motion. For uniformly *retarded* motion, j is a negative number.

If the spaces be counted from the position of the moving point at the time $t = 0$, we have $s_0 = 0$, and the equations become

$$v = v_0 + jt, \quad s = v_0t + \tfrac{1}{2}jt^2, \quad \tfrac{1}{2}(v^2 - v_0^2) = js.$$

If in addition the initial velocity v_0 be zero, the point starting from rest at the time $t = 0$, the equations reduce to the following:

$$v = jt, \quad s = \tfrac{1}{2}jt^2, \quad \tfrac{1}{2}v^2 = js.$$

17. The most important example of uniformly accelerated motion is furnished by a body falling in vacuo near

the earth's surface. Assuming that the body does not rotate during its fall, its motion relative to the earth is a mere *translation*, *i. e.* the velocities of all its points are equal and parallel; and it is sufficient to consider the motion of any one point of the body. It is known from observation and experiment that under these circumstances the acceleration of a falling body is constant at any given place and equal to about 980 cm., or 32.2 ft., per second per second; the value of this so-called *acceleration of gravity* is usually denoted by g.

In the exercises on falling bodies (Art. 19) we make throughout the following simplifying assumptions: the falling body does not rotate; the resistance of the air is neglected, or the body falls in vacuo; the space fallen through is so small that g may be regarded as constant; the earth is regarded as fixed.

18. The velocity v acquired by a falling body after falling from rest through a height h is found from the last equation of Art. 16 as

$$v = \sqrt{2gh}.$$

This is usually called the **velocity due to the height** (or **head**) h, while $h = v^2/2g$ is called the **height** (or **head**) **due to the velocity** v.

19. Exercises.

(1) A body falls from rest at a place where $g = 32.2$. Find (a) the velocity at the end of the fourth second; (b) the space fallen through in 4 seconds; (c) the space fallen through in the fifth second.

(2) A train, starting from the station, acquires a velocity of 30 M./h.: (a) in 8 min.; (b) in 2 miles; what was its acceleration (regarded as constant)?

(3) Galileo, who first discovered the laws of falling bodies, expressed them in the following form: (a) The velocities acquired at the end of the successive seconds increase as the natural numbers; (b) the spaces described during the successive seconds increase as the

odd numbers; (c) the spaces described from the beginning of the motion to the end of the successive seconds increase as the squares of the natural numbers. Prove these statements.

(4) A stone dropped into the vertical shaft of a mine is heard to strike the bottom after t seconds; find the depth of the shaft, if the velocity of sound be given $= c$. Assume $t = 4$ s., $c = 332$ meters, $g = 980$.

(5) A railroad train in approaching a station makes half a mile in the first, 2,000 ft. in the second, minute of its retarded motion. If the motion is *uniformly* retarded: (a) When will it stop? (b) What is the retardation? (c) What was the initial velocity? (d) When will the velocity be 4 miles an hour?

(6) A body being projected vertically upwards with an initial velocity v_0, (a) how long and (b) to what height will it rise? (c) When and (d) with what velocity does it reach the starting-point?

(7) A bullet is shot vertically upwards with an initial velocity of 1200 ft. per second. (a) How high will it ascend? (b) What is its velocity at the height of 16,000 ft.? (c) When will it reach the ground again? (d) With what velocity? (e) At what time is it 16,000 ft. above the ground? Explain the meaning of the double sign in (e). Use $g = 32$.

(8) With what velocity must a ball be thrown vertically upwards to reach a height of 100 ft.?

(9) A body is dropped from a point B at a height $AB = h$ above the ground; at the same time another body is thrown vertically upward from the point A, with an initial velocity v_0. (a) When and (b) where will they collide? (c) If they are to meet at the height $\frac{1}{2}h$, what must be the initial velocity?

(10) The barrel of a rifle is 30 in. long; the muzzle velocity is 1300 ft./sec.; if the motion in the barrel be uniformly accelerated, what is the acceleration and what the time?

(11) If a stone dropped from a balloon while ascending at the rate of 25 ft./sec. reaches the ground in 6 seconds, what was the height of the balloon when the stone was dropped?

(12) If the speed of a train increases uniformly after starting for 8 minutes while the train travels 2 miles, what is the velocity acquired?

(13) Two particles fall from rest from the same point, at a short interval of time τ; find how far they will be apart when the first par-

ticle has fallen through a height h. Take *e. g.*, $h = 900$ ft., $\tau = \frac{1}{100}$ second.

20. Acceleration inversely proportional to the square of the distance, *i. e.* $j = \mu/s^2$ where μ is a constant (viz. the acceleration at the distance $s = 1$) and s is the distance of the moving point from a fixed point in the line of motion.

The differential equation (1) becomes in this case

$$\frac{d^2s}{dt^2} = \frac{\mu}{s^2}; \tag{5}$$

the first integration is readily performed by multiplying both members by ds/dt so as to make the left-hand member the exact derivative of $\frac{1}{2}(ds/dt)^2$ or $\frac{1}{2}v^2$. Thus we find

$$\tfrac{1}{2}v^2 = \mu\int\frac{ds}{s^2} = -\frac{\mu}{s} + C, \tag{6}$$

where the constant of integration, C, must be determined from the so-called initial conditions of the problem. For instance, if $v = v_0$ when $s = s_0$, we have $\frac{1}{2}v_0^2 = -\mu/s_0 + C$; hence, eliminating C between this relation and (6),

$$\tfrac{1}{2}(v^2 - v_0^2) = -\mu\left(\frac{1}{s} - \frac{1}{s_0}\right). \tag{7}$$

To perform the second integration solve this equation for v and substitute ds/dt for v:

$$\frac{ds}{dt} = \pm\sqrt{v_0^2 + \frac{2\mu}{s_0} - \frac{2\mu}{s}},$$

or putting $v_0^2 + 2\mu/s_0 = 2\mu/\mu'$,

$$\frac{ds}{dt} = \pm\sqrt{\frac{2\mu}{\mu'}}\cdot\sqrt{\frac{s-\mu'}{s}}. \tag{8}$$

Here the variables s and t can be separated, and we find if $s = s_0$ for $t = 0$

$$t = \pm \sqrt{\frac{\mu'}{2\mu}} \int_{s_0}^{s} \sqrt{\frac{s}{s - \mu'}}\, ds. \tag{9}$$

To integrate put $s - \mu' = x^2$. The result will be different according to the signs of μ, μ', and v, which must be determined from the nature of the particular problem.

It is easily seen that the methods of integration used in this problem apply whenever j is given as a function of s alone.

21. Whenever in nature we observe a motion not to remain uniform, we try to account for the change in the character of the motion by imagining a special cause for such change. In rectilinear motion, the only change that can occur in the motion is a change in the velocity, *i. e.*, an acceleration (or retardation). It is often convenient to have a special name for this supposed cause producing acceleration or retardation; we call it **force** (attraction, repulsion, pressure, tension, friction, resistance of a medium, elasticity, cohesion, etc.), and assume it to be proportional to the acceleration A fuller discussion of the nature of force and its relation to mass will be found in Arts. 171–188. The present remark is only intended to make more intelligible the physical meaning and application of the problems to be discussed in the following articles.

22. It is an empirical fact that the acceleration of bodies falling in vacuo on the earth's surface is constant only for distances from the surface that are very small in comparison with the radius of the earth. For larger distances the acceleration is found inversely proportional to the square of the distance from the earth's center.

By a bold generalization Newton assumed this law to hold generally between any two particles of matter, and this assumption has been verified by subsequent observations. It can therefore be regarded as a general law of nature that any particle of matter produces in every other such particle, each particle being regarded as concentrated at a point, an acceleration inversely proportional to the square of the distance between these points. This is known as *Newton's law of universal gravitation*, the acceleration being regarded as caused by a force of attraction inherent in each particle of matter.

It is shown in the theory of attraction (Art. 253) that the attraction of a spherical mass, such as the earth, on any particle *outside* the sphere

is the same as if the mass of the sphere were concentrated at its center. The acceleration produced by the earth on any particle outside it is therefore inversely proportional to the square of the distance of the particle from the center of the earth.

23. Let us now apply the general equations of Art. 20 to the particular case of a body falling from a great height towards the center of the earth, the resistance of the air being neglected.

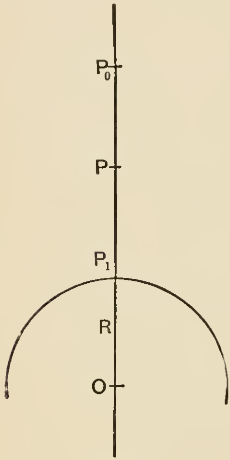

Fig. 3.

Let O be the center of the earth (Fig. 3), P_1 a point on its surface, P_0 the initial position of the moving point at the time $t = 0$, P its position at the time t; let $OP_1 = R$, $OP_0 = s_0$, $OP = s$; and let g be the acceleration at P_1, j the acceleration at P, both in absolute value. Then, according to Newton's law, $j : g = R^2 : s^2$. This relation serves to determine the value of the constant μ in (5); for since the acceleration is to have the value g when $s = R$ we have

$$\left(\frac{d^2s}{dt^2}\right)_{s=R} = \frac{\mu}{R^2} = -g,$$

the minus sign being taken because the acceleration is directed toward the origin O. We have therefore

$$\mu = -gR^2,$$

so that (5) becomes in our case

$$\frac{d^2s}{dt^2} = -\frac{gR^2}{s^2}, \tag{10}$$

the minus sign indicating that the acceleration tends to diminish the distances counted from O as origin.

The integration can now be performed as in Art. 20. Multiplying by ds/dt and integrating, we find $\frac{1}{2}v^2 = gR^2/s + C$. If the initial velocity be zero, we have $v = 0$ for $s = s_0$; hence $C = -gR^2/s_0$, and

$$v = -R\sqrt{2g}\sqrt{\frac{1}{s} - \frac{1}{s_0}} = -R\sqrt{\frac{2g}{s_0}}\sqrt{\frac{s_0 - s}{s}}. \tag{11}$$

Here again the minus sign before the radical is selected since the velocity v is directed in the sense opposite to that of the distance s.

Substituting ds/dt for v and separating the variables t and s we have

$$dt = -\frac{1}{R}\sqrt{\frac{s_0}{2g}}\sqrt{\frac{s}{s_0 - s}}\,ds;$$

hence, integrating as indicated at the end of Art. 20:

$$t = \frac{1}{R}\sqrt{\frac{s_0}{2g}}\left(\sqrt{s(s_0 - s)} + s_0 \sin^{-1}\sqrt{\frac{s_0 - s}{s_0}}\right),$$

the constant of integration being zero since $s = s_0$ for $t = 0$. The last term can be slightly simplified by observing that

$$\sin^{-1}\sqrt{1 - u^2} = \cos^{-1}u,$$

whence finally:

$$t = \frac{1}{R}\sqrt{\frac{s_0}{2g}}\left(\sqrt{s(s_0 - s)} + s_0 \cos^{-1}\sqrt{\frac{s}{s_0}}\right). \tag{12}$$

24. Exercises.

(1) Find the velocity with which the body arrives at the surface of the earth if it be dropped from a height equal to the earth's radius, and determine the time of falling through this height. Take $R = 4000$ miles, $g = 32$.

(2) Show that formula (11) reduces to $v = \sqrt{2gh}$ (Art. 18) with $s = R$ if $s_0 - s = h$ is small in comparison with R.

(3) Show that when s_0 is large in comparison with R while s differs but slightly from R, the formulæ (11) and (12) reduce approximately to

$$v = -\sqrt{2g\frac{R}{s}}, \quad t = \frac{\pi s_0^{\frac{3}{2}}}{2\sqrt{2gR}}.$$

Hence find the final velocity and time of fall of a body falling to the earth's surface (*a*) from an infinite distance; (*b*) from the moon ($s_0 = 60\,R$).

(4) Derive the expressions for v and t corresponding to (11) and (12) when the initial velocity is v_0 (toward the center).

(5) A particle is projected vertically upwards from the earth's surface with an initial velocity v_0. How far and how long will it rise?

(6) If, in (5), the initial velocity be $v_0 = \sqrt{gR}$, how high and how long will the particle rise? How long will it take the particle to rise and fall back to the earth's surface?

25. Acceleration directly proportional to the distance, *i. e* $j = \kappa s$, where κ is a constant.

The equation of motion

$$\frac{d^2s}{dt^2} = \kappa s \qquad (13)$$

can be integrated by the method used in Art. 20. The result of the second integration will again be different according to the sign of κ. We shall study here only a special case, reserving the general discussion of this law of acceleration until later.

26. It is shown in the theory of attraction (Art. 251) that the attraction of a spherical mass such as the earth on any point *within* the mass produces an acceleration directed to the center of the sphere and proportional to the distance from this center. Thus, if we imagine a particle moving along a diameter of the earth, say in a straight narrow tube

passing through the center, we should have a case of the motion represented by equation (13).

To determine the value of κ for our problem we notice that at the earth's surface, that is, at the distance $OP_1 = R$ from the center O (Fig. 4), the acceleration must be g. If, therefore, j denote the numerical value of the acceleration at any distance $OP = s(< R)$, we have $j : g = s : R$, or $j = gs/R$. But the acceleration tends to diminish the distance s, hence $d^2s/dt^2 = -(g/R)s$. Denoting the positive constant g/R by μ^2, the equation of motion is

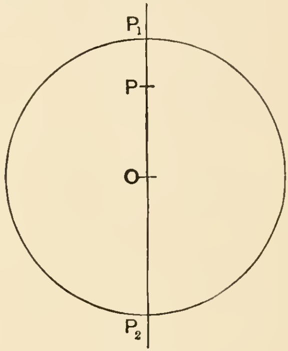

Fig. 4.

$$\frac{d^2s}{dt^2} = -\mu^2 s, \text{ where } \mu = \sqrt{\frac{g}{R}}. \tag{14}$$

Integrating as in Arts. 20 and 23, we find

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\mu^2 s^2 + C.$$

If the particle starts from rest at the surface, we have $v = 0$ when $s = R$; hence $0 = -\frac{1}{2}\mu^2 R^2 + C$; and subtracting this from the preceding equation, we find

$$v = -\mu\sqrt{R^2 - s^2}, \tag{15}$$

where the minus sign of the square root is selected because s and v have opposite sense.

Writing ds/dt for v and separating the variables, we have

$$dt = -\frac{1}{\mu}\frac{ds}{\sqrt{R^2 - s^2}},$$

whence

$$t = \frac{1}{\mu} \cos^{-1} \frac{s}{R} + C'.$$

As $s = R$ when $t = 0$, we have $0 = \frac{1}{\mu} \cos^{-1} 1 + C'$, or $C' = 0$. Solving for s, we find

$$s = R \cos\mu t. \tag{16}$$

Differentiating, we obtain v in terms of t:

$$v = -\mu R \sin\mu t. \tag{17}$$

27. The motion represented by equations (16) and (17) belongs to the important class of *simple harmonic motions* (see Arts. 71 sq.). The particle reaches the center when $s = 0$, *i. e.* when $\mu t = \pi/2$, or at the time $t = \pi/2\mu$. At this time the velocity has its maximum value. After passing through the center the point moves on to the other end, P_2, of the diameter, reaches this point when $s = -R$, *i. e.* when $\mu t = \pi$, or at the time $t = \pi/\mu$. As the velocity then vanishes, the moving point begins the same motion in the opposite sense.

The time of performing one complete oscillation (back and forth) is called the **period** of the simple harmonic motion; it is evidently

$$T = 4 \cdot \frac{\pi}{2\mu} = \frac{2\pi}{\mu}.$$

28. Exercises.

(1) Equation (14) is a differential equation whose general integral is known to be of the form

$$s = C_1 \sin\mu t + C_2 \cos\mu t;$$

determine the constants C_1, C_2 and deduce equations (16) and (17).

(2) Find the velocity at the center and the period, taking $g = 32$ and $R = 4000$ miles.

(3) A point whose acceleration is proportional to its distance from a fixed point O starts at the distance s_0 from O with a velocity v_0 directed away from O; how far will it go before returning?

CHAPTER II.

TRANSLATION AND ROTATION.

29. In kinematics, the term **rigid body** is used to denote a figure of invariable size and shape, or an aggregate of points whose distances from each other remain unchanged.

The position of a rigid body is given if the positions of any three of its points, not in a straight line, are given; when three such points are fixed the body is fixed.

The kinematics of rigid bodies will be discussed more fully later on (Arts. 114–150); it will here suffice to mention two particular types of motion of a rigid body: translation, and rotation about a fixed axis.

30. The motion of a rigid body is called a **translation** when all points of the body describe equal and parallel curves. This will be the case if any three points of the body, not in a straight line, describe equal and parallel curves. Owing to the *rigidity* of the body, *i. e.* the invariability of the mutual distances of its points, the velocities and accelerations of all points at any given instant must then be equal; thus, *in translation, the motion of the whole body is given by the motion of any one of its points.*

31. When a rigid body has two of its points fixed, the only motion it can have is a **rotation** about the line joining the fixed points as axis. Thus, *in rotation about a fixed axis, all points of the body excepting those on the axis describe arcs of circles whose centers lie on the axis and whose planes are perpendicular to the axis*; all points on the axis are at rest.

The position of a rigid body with a fixed axis l is given by

the position of any one of its points P, not on the axis. This position is most conveniently assigned by the dihedral angle θ, made by the plane (l, P) of the body with a fixed plane through l. If a definite sense of rotation about the axis is assumed as positive, say the counter-clockwise sense as seen from a marked end of the axis (Fig. 5), the angle θ, expressed in radians, is a real number and serves as *co-ordinate* to determine the position of the body.

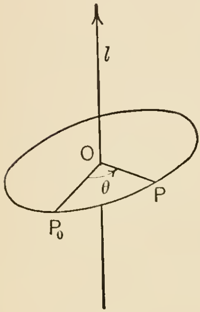

Fig. 5.

32. As the body turns about the axis l in any way, the angle θ varies with the time; the co-ordinate θ can be regarded as a function of the time, just as in the case of the rectilinear motion of a point (and hence (Art. 30) also in the rectilinear translation of a rigid body) the co-ordinate s is a function of the time.

The rotation is called *uniform* if equal angles are always described in equal times. In this case the quotient θ/t of the angle θ described in any time t, divided by this time, is called the *angular velocity*, ω, *of the uniform rotation*:

$$\omega = \frac{\theta}{t}.$$

If, in particular, the time of a complete revolution be denoted by T, we have for uniform rotation:

$$\omega = \frac{2\pi}{T}.$$

In the applications, angular velocity is often measured by the number of complete revolutions per unit of time. Thus, if n be the number of revolutions per second, N that per minute, we have

$$\omega = 2\pi n = \frac{\pi N}{30}.$$

33. When the rotation is *not* uniform, the quotient obtained by dividing the angle of rotation by the time in which it is described, gives the *mean*, or *average, angular velocity* for that time.

The rate of change of the angle of rotation with the time at any particular moment is called the **angular velocity** *at that moment*:

$$\omega = \frac{d\theta}{dt}.$$

The rate at which the angular velocity changes with the time is called the **angular acceleration**; denoting it by α, we have

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

34. The most important special case of variable angular velocity is that of uniformly accelerated (or retarded) rotation when the angular acceleration is constant. The formulæ for this case have precisely the same form as those given in Arts. 14–16 for uniformly accelerated rectilinear motion. Denoting the constant linear acceleration by j, we have, when the initial velocity is 0,

FOR TRANSLATION:	FOR ROTATION:
$\frac{d^2s}{dt^2} = j$, a constant;	$\frac{d^2\theta}{dt^2} = \alpha$, a constant;
$v = jt$,	$\omega = \alpha t$,
$s = \frac{1}{2}jt^2$,	$\theta = \frac{1}{2}\alpha t^2$,
$\frac{1}{2}v^2 = js$;	$\frac{1}{2}\omega^2 = \alpha\theta$;

and when the initial velocities are v_0 and ω_0, respectively:

FOR TRANSLATION:	FOR ROTATION:
$v = v_0 + jt,$	$\omega = \omega_0 + \alpha t,$
$s = v_0 t + \frac{1}{2}jt^2,$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2,$
$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = js;$	$\frac{1}{2}\omega^2 - \frac{1}{2}\omega_0^2 = \alpha\theta.$

35. Let a point P, whose perpendicular distance from the axis of rotation is $OP = r$, rotate about the axis with the angular velocity $\omega = d\theta/dt$. In the element of time, dt, it will describe an element of arc $ds = rd\theta = r\omega dt$. Its velocity $v = ds/dt$ (frequently called its **linear** velocity to distinguish it from the angular velocity) is therefore related to the angular velocity of rotation by the equation

$$v = \omega r.$$

The close analogy between rectilinear translation and rotation about a fixed axis, which is not confined to uniform or uniformly accelerated motion and arises from the fact that in each of the two cases the position of the body is determined by a single co-ordinate, can be illustrated by laying off on the axis of rotation a length measuring the angle of rotation. The rectilinear motion of the extremity of this vector along the axis gives an exact representation of the rotation.

36. Exercises.

(1) If a fly-wheel of 10 ft. diameter makes 30 revolutions per minute, what is its angular velocity, and what is the linear velocity of a point on its rim?

(2) Find the constant acceleration (such as the retardation caused by a Prony brake) that would bring the fly-wheel in Ex. (1) to rest in ⅓ minute. How many revolutions does the fly-wheel make during its retarded motion before it comes to rest?

(3) A wheel is running at a uniform speed of 32 turns a second when a resistance begins to retard its motion uniformly at a rate of 8 radians

per second. (*a*) How many turns will it make before stopping? (*b*) In what time is it brought to rest?

(4) A wheel of 6 ft. diameter is making 50 rev./min. when thrown out of gear. If it comes to rest in 4 minutes, find (*a*) the angular retardation; (*b*) the linear velocity of a point on the rim at the beginning of the retarded motion; (*c*) the same after two minutes.

CHAPTER III.

CURVILINEAR MOTION OF A POINT.

1. Relative velocity; composition and resolution of velocities.

37. It is often convenient to think of the velocity of a point not as a mere number, but as a **vector**, *i. e.* a segment PQ of a straight line (Fig. 6), drawn from the point P in the direction of motion and representing by its length the magnitude of the velocity, by its direction the direction of motion of P, and by an arrowhead the sense of the motion.

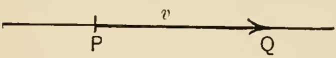

Fig. 6.

38. Consider a point P (Fig. 7) moving along a straight line l with constant velocity v_r, while the line l moves in a fixed plane with a constant velocity v_b in a direction making an angle α with the line l. Then the vector $PQ = v_r$ is called the **relative velocity** of P with respect to l; the vector $PS = v_b$ may be called the *body velocity*, or the velocity of the body of reference (here the line l).

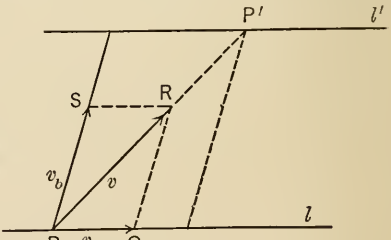

Fig. 7.

With respect to the fixed plane, the point P has not only the velocity v_r, but it participates in the motion of l. Its *absolute velocity* v, *i. e.* its velocity with respect to the fixed plane, is therefore represented in magnitude, direction, and

sense by the vector PR, *i. e.* by the diagonal of the parallelogram constructed on the vectors v_r and v_b. This vector $PR = v$ is called the *resultant*, or *geometric sum*, of the vectors $PQ = v_r$ and $PS = v_b$.

It is easy to see that this result will hold even when the motions are not uniform, provided we mean by v_r the instantaneous relative velocity of P and by v_b the simultaneous velocity of that point of the body of reference with which P happens to coincide at the instant.

We have thus the general proposition that *the absolute velocity v of a point P is the resultant, or geometric sum, of its relative velocity v_r and the body velocity v_b.*

39. The term "geometric sum," of the vectors $v_r = PQ$ and $v_b = PS$ may be justified by observing that (Fig. 7) $QR = PS$; hence the resultant $PR = v$ is obtained simply by adding the vectors v_r and v_b geometrically, *i. e.* by drawing first the vector $PQ = v_r$ and then from its extremity Q the vector $QR = v_b$.

Conversely, *the relative velocity $PQ = v_r$ is found by geometrically subtracting the body velocity v_b from the absolute velocity v*; *i. e.* by drawing the vector $PR = v_r$ and from its extremity R the vector RQ equal and opposite to the vector $PS = v_b$. This result can be interpreted as follows: In the example of Art. 38 of a point moving with velocity v_r along the line l while l moves with velocity v_b in a fixed plane, let us superimpose the velocity $-v_b$, *i. e.* a velocity equal and opposite to the body velocity, on the whole system, formed by the line and the point; the line is thereby brought to rest while the point will have the velocities v and $-v_b$ whose resultant is the relative velocity v_r. Hence *the relative velocity is found as the resultant of the absolute velocity and the body velocity reversed.*

40. It is this idea of relative motion that leads to the so-called *parallelogram of velocities, i. e.* to the proposition that a point whose velocity is $v = PR$ (Fig. 8) can be regarded as possessing simultaneously any two velocities, such as $v_1 = PQ$, $v_2 = PS = QR$, whose geometric sum is $v = PR$. For we can always regard v_1 as the relative velocity of the point along the line PQ and v_2 as the body velocity, *i. e.* as the velocity of the line PQ.

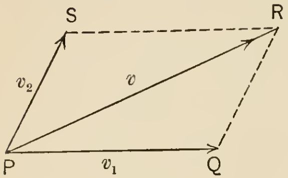

Fig. 8.

41. Finally, if in the example of Art. 38 we suppose the plane π in which the line l moves to have itself a velocity v_π,

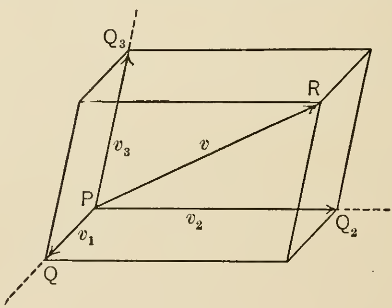

Fig. 9.

it is clear that the absolute velocity v of the point will be the resultant, or geometric sum, of the three velocities v_r, v_b, v_π; *i. e.* it will be represented by the diagonal of the parallelepiped that has the vectors v_r, v_b, v_π as adjacent edges. It then follows that the velocity $v = PR$ of a point (Fig. 9) can be regarded as equivalent to any three simultaneous velocities $v_1 = PQ_1$, $v_2 = PQ_2$, $v_3 = PQ_3$, whose geometric

sum is $v = PR$. This proposition is known as the *parallelepiped of velocities*; v_1, v_2, v_3 are called the **components** of v.

The corresponding propositions for forces in statics will be familiar to the student from elementary physics. But it will be seen later that these propositions in statics are really based on the more elementary propositions for velocities.

42. Exercises.

(1) The components of the velocity of a point are 5 and 3 ft./sec. and enclose an angle of 135°; find the resultant in magnitude and direction. Check the result by graphical construction.

(2) Find the components of a velocity of 10 ft./sec., along two lines inclined to it at 30° and 90°.

(3) A man jumps from a car at an angle of 60°, with a velocity of 9 ft./sec. (relatively to the car). If the car is running 10 M./h., with what velocity and in what direction does the man strike the ground?

(4) Two men, A and B, walking at the rate of 3 and 4 M./h., respectively, cross each other at a rectangular street corner. Find the relative velocity of A with respect to B in magnitude and direction.

(5) How must a man throw a stone from a train running 15 M./h. to make it move 10 ft./sec. at right angles to the track?

(6) The velocity of light being 300,000 km./sec., the velocity of the earth in its orbit 30 km./sec., determine approximately the constant of the aberration of the fixed stars.

(7) A man on a wheel, riding along the railroad track at the rate of 9 M./h., observes that a train meeting him takes 3 sec. to pass him, while a train of equal length takes 5 sec. to overtake him. If the trains have the same speed, what is it? What is the length of the train?

(8) A swimmer starting from a point A on one bank of a river wishes to reach a certain point B on the opposite bank. The velocity v_b of the current and the angle $\theta (< \frac{1}{2}\pi)$ made by AB with the current being given, determine the least relative velocity v_r of the swimmer in magnitude and direction.

(9) A straight line in a plane turns with constant angular velocity ω about one of its points O, while a point P, starting from O, moves along the line with constant velocity v_0. Determine the absolute path of P and its absolute velocity v.

(10) Show how to construct the tangent and normal to the spiral of Archimedes, $r = a\theta$, where $\theta = \omega t$.

2. Velocity in curvilinear motion.

43. If on the curve described by the moving point we select an origin P_0, and take a definite sense of progression along the curve as positive, the position P of the point at any time t is given by the arc $P_0P = s$, which might be regarded as the co-ordinate of P (Fig. 10).

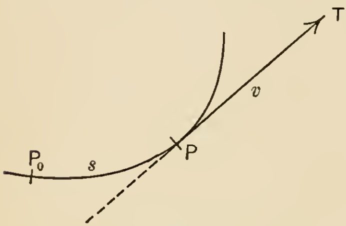

Fig. 10.

As s is a function of the time t, its time-derivative

$$v = \frac{ds}{dt}$$

gives the magnitude of the velocity of the point at P, or at the instant t, in its curvilinear motion (comp. Art. 5).

To incorporate in the definition of velocity the idea of the varying *direction* of the motion, which at any instant t is that of the tangent to the path, we lay off from P, on this tangent, a segment PT of length $v = ds/dt$, in the sense of the motion, and define the vector PT as the **velocity** of the point in its curvilinear motion (comp. Art. 37).

44. When the motion of the point P is referred to fixed rectangular axes Ox, Oy, Oz, the co-ordinates x, y, z of P (Fig. 11) are functions of the time:

$$x = x(t),\ y = y(t),\ z = z(t).$$

Now the x-co-ordinate of P is at the same time the co-ordinate of the projection P_x of P on the axis Ox on this axis. As the point P moves in space, its projection P_x moves

along the axis Ox, and the velocity of P_x in its rectilinear motion is

$$v_x = \frac{dx}{dt}.$$

Similarly the velocities of the projections P_y, P_z of P on Oy, Oz are

$$v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}.$$

The rectilinear motions of P_x, P_y, P_z along the axes Ox, Oy, Oz, respectively, fully determine the curvilinear motion of $P(x, y, z)$ in space.

45. On the other hand, the velocity-vector $PT = v$ can, by Art. 41, be resolved into its three components along the

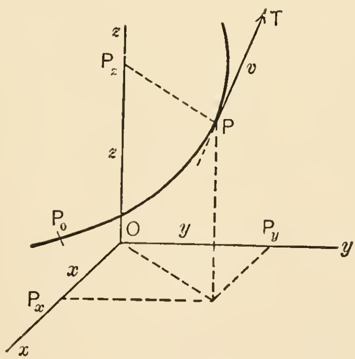

Fig. 11.

axes; if the tangent to the path at P makes the angles α, β, γ with Ox, Oy, Oz, respectively, these components are

$$v \cos\alpha, \quad v \cos\beta, \quad v \cos\gamma.$$

It is easy to show that *these components of v are equal, respectively, to the velocities dx/dt, dy/dt, dz/dt of the projections*

P_x, P_y, P_z *of* P *on the axes.* For we have, if Δs is the arc described by P in the time Δt:

$$\frac{dx}{dt} = \lim_{\Delta t=0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t=0} \frac{\Delta x}{\Delta s}\frac{\Delta s}{\Delta t} = \cos\alpha \frac{ds}{dt} = v \cos\alpha,$$

since at any ordinary point of the curve (*i. e.* at any point at which the curve possesses a definite tangent) we have

$$\lim \frac{\Delta x}{\Delta s} = \cos\alpha.$$

Similarly for dy/dt, dz/dt.

We shall therefore henceforth denote by v_x, v_y, v_z not only (as in Art. 44) the velocities of P_x, P_y, P_z, but also the components of the velocity v along the axes Ox, Oy, Oz. Thus we have:

$$v_x = v \cos\alpha = \frac{dx}{dt}, \; v_y = v \cos\beta = \frac{dy}{dt}, \; v_z = v \cos\gamma = \frac{dz}{dt},$$

$$v^2 = v_x^2 + v_y^2 + v_z^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2.$$

In the language of infinitesimals we may say that the velocity is found by dividing the element of arc $ds = \sqrt{dx^2 + dy^2 + dz^2}$ by dt.

46. *In polar co-ordinates* $OP = r$, $xOP = \theta$, $yOQ = \phi$ (Fig. 12), the rectangular components v_r, v_θ, v_ϕ of the velocity v along OP, at right angles to OP in the plane xOP, and at right angles to this plane are readily found from the last remark in Art. 45, by observing that

$$ds^2 = dr^2 + (rd\theta)^2 + (r \sin\theta \, d\phi)^2,$$

whence

$$v_r = \frac{dr}{dt}, \; v_\theta = r\frac{d\theta}{dt}, \; v_\phi = r \sin\theta \frac{d\phi}{dt}.$$

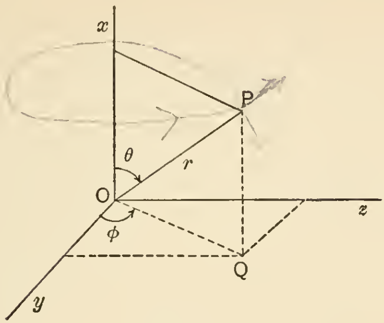

Fig. 12.

47. If the path of P is a *plane* curve we have in rectangular cartesian co-ordinates

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2};$$

and in polar co-ordinates

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt}, \quad v = \frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}.$$

As the point P moves in the plane curve its radius vector OP sweeps out the polar area S of the curve, *i. e.* the area bounded by any two radii vectores and the arc of the curve between their ends. If ΔS be the increment of this area in the time Δt, the limit of the ratio $\Delta S/\Delta t$, as Δt approaches zero, is called the **sectorial velocity** dS/dt of the point P (about the origin O):

$$\frac{dS}{dt} = \lim_{\Delta t = 0} \frac{\Delta S}{\Delta t}.$$

It follows from the well-known expression for the element of polar area that in polar co-ordinates

$$\frac{dS}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt},$$

and in rectangular cartesian co-ordinates

$$\frac{dS}{dt} = \tfrac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

48. Exercises.

(1) If the point P describes a *circle* of radius a about the origin O, with angular velocity ω, the linear velocity of P is $v = a\omega$ (Art. 35); its components along rectangular axes through the origin are (Fig. 13):

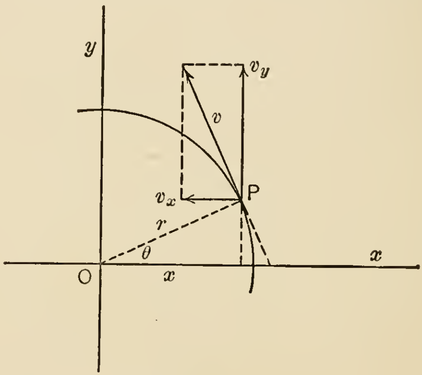

Fig. 13.

$$v_x = a\omega\cos(\tfrac{1}{2}\pi + \theta) = -a\omega\sin\theta = -\omega y,$$

$$v_y = a\omega\sin(\tfrac{1}{2}\pi + \theta) = a\omega\cos\theta = \omega x.$$

Obtain these results by differentiating the equations of the circle $x = a\cos\theta$, $y = a\sin\theta$ with respect to the time.

(2) Show that the velocity of a point describing a *cycloid* passes through the highest point of the generating circle.

(3) The *ellipse* being defined as the locus of a point such that the sum of its distances from two fixed points is constant, show that the normal bisects the angle between the focal radii r_1, r_2.

In bilinear co-ordinates the equation of the ellipse is simply

$$r_1 + r_2 = 2a.$$

Differentiating with respect to t and denoting time-derivatives by dots, we find

$$\dot{r}_1 + \dot{r}_2 = 0;$$

i. e. the rate of increase of one focal radius is equal to the rate of decrease of the other. Notice, however, that $\dot{r}_1$ and $\dot{r}_2$ are not the *components* of the velocity of the describing point P along the focal radii, but the *projections* of this velocity on these radii. For, the velocity v of P can be resolved: (a) into $\dot{r}_1$ along r_1 and a component perpendicular to r_1; (b) into $\dot{r}_2$ along r_2 and a component perpendicular to r_2. Both resolutions arise from the same vector v; hence perpendiculars erected at the extremities of $\dot{r}_1$ and $\dot{r}_2$ (laid off from P along r_1, r_2 in the proper sense) must meet at the extremity of v. As $\dot{r}_2 = -\dot{r}_1$, v bisects the angle between r_1 (produced) and r_2.

(4) Find a construction for the tangent to any conic given by directrix, focus, and eccentricity.

(5) Derive the expressions for v_r and v_θ in Art. 46 by the method of limits.

3. Acceleration in curvilinear motion.

49. As the moving point describes its path the velocity vector $v = PT$ (Art. 43) will in general vary both in magnitude and in direction. To compare the velocities $v = PT$ at the time t and $v' = P'T'$ at the time $t + \Delta t$ (Fig. 14)

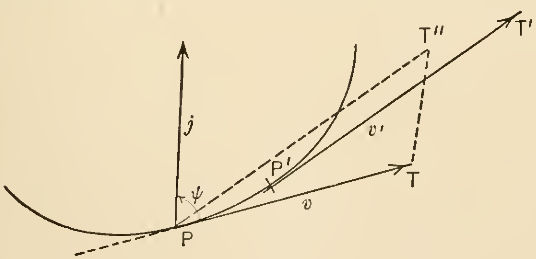

Fig. 14.

we must draw these vectors from the same origin, say from the point P. Making $PT'' = P'T' = v'$, it appears that

the vector v' can be obtained from the vector v by adding to it geometrically the vector TT'' which represents the geometrical increment of the velocity in the time interval Δt.

This vector TT'', divided by Δt, is the *average acceleration* in the time Δt. As Δt approaches zero, the vector TT'' approaches zero; but its direction will in general approach a definite direction as a limit, and the ratio of its length to Δt will approach a definite number as limit. A vector (generally drawn from the point P) having this limiting direction as its direction and a length

$$j = \lim_{\Delta t = 0} \frac{TT''}{\Delta t}$$

is defined as the **acceleration** of the moving point at P, or at the time t.

It follows from this definition that the acceleration vector lies in the osculating plane of the path at P, this plane being the limiting position of the plane determined by the tangent at P and any near point P' of the curve as P' approaches P along the curve.

50. Acceleration being defined as a vector can be resolved into components by the parallelogram or parallelepiped rules (Arts. 40, 41).

Thus, in particular, the acceleration j, since it lies in the osculating plane, can be resolved into a *tangential component* j_t along the tangent, and a *normal component* j_n along the principal normal at P, the principal normal being the intersection of the normal plane with the osculating plane. If ψ (Fig. 14) is the angle between the velocity and the acceleration these components are

$$j_t = j \cos\psi, \quad j_n = j \sin\psi.$$

51. If from any fixed point O we draw vectors OQ equal and parallel to the velocity vectors PT of the moving point P, the extremities Q lie on a curve called the **hodograph** of the path of P; and it follows from Art. 49 that *the acceleration vector of P is equal and parallel to the velocity vector in the motion of Q along the hodograph.* Hence the tangential and normal components of the acceleration of P are equal, respectively, to the components of the velocity of Q along the radius vector OQ and at right angles to it. Observing that the acceleration lies in the osculating plane we have therefore by Art. 47

$$j_t = \frac{dv}{dt}, \quad j_n = v\frac{d\theta}{dt},$$

where θ is the angle made by OQ, *i. e.* by the velocity vector at P, with any fixed direction in the osculating plane. Now if ds be the element of arc of the path of P we have (comp. below, Art. 54)

$$\frac{d\theta}{ds} = \frac{1}{\rho},$$

where ρ is the radius of (first) curvature of the path at P; hence

$$j_n = v\frac{d\theta}{ds}\frac{ds}{dt} = \frac{v^2}{\rho}.$$

Thus we have for the **tangential acceleration** j_t and the **normal acceleration** j_n of a moving point

$$j_t = \frac{dv}{dt}, \quad j_n = \frac{v^2}{\rho}.$$

52. When the rectangular cartesian co-ordinates of the moving point are given as functions of the time,

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

their first derivatives with respect to the time are on the

one hand the velocities of the projections P_x, P_y, P_z of P on the axes in their rectilinear motions, on the other the components v_x, v_y, v_z of the velocity $v = ds/dt$ of P in its curvilinear motion (Art. 44). Thus, using dots to denote time-derivatives, we have

$$v_x = \dot{x},\ v_y = \dot{y}\ v_z = \dot{z}.$$

It will now be shown that *the second time-derivatives* $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ of x, y, z, which are the accelerations of P_x, P_y, P_z in their rectilinear motions, *are at the same time the components* j_x, j_y, j_z *of the acceleration vector along the axes of co-ordinates.*

53. We have

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = v\frac{dx}{ds},$$

whence, differentiating with respect to t,

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(v\frac{dx}{ds}\right) = \frac{dv}{dt}\frac{dx}{ds} + v\frac{d^2x}{ds^2}\frac{ds}{dt} = \dot{v}\frac{dx}{ds} + v^2\frac{d^2x}{ds^2}.$$

Writing down the corresponding expressions for $\ddot{y}$, $\ddot{z}$ by cyclic permutation of x, y, z we find:

$$\ddot{x} = \dot{v}\frac{dx}{ds} + v^2\frac{d^2x}{ds^2},$$

$$\ddot{y} = \dot{v}\frac{dy}{ds} + v^2\frac{d^2y}{ds^2},$$

$$\ddot{z} = \dot{v}\frac{dz}{ds} + v^2\frac{d^2z}{ds^2}.$$

Now if α, β, γ are the direction cosines of the velocity vector we have

$$\alpha = \frac{dx}{ds},\ \beta = \frac{dy}{ds},\ \gamma = \frac{dz}{ds};$$

hence the first terms in the expressions found for $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are

the components along the axes of a vector, parallel to the velocity and of length $\dot{v}$, *i. e.* of the tangential acceleration j_t (Art. 51).

To see that the second terms are the components of the normal acceleration $j_n = v^2/\rho$ (Art. 51) we have only to remember that the direction cosines λ, μ, ν of the principal normal of any curve are

$$\lambda = \rho \frac{d^2x}{ds^2}, \quad \mu = \rho \frac{d^2y}{ds^2}, \quad \nu = \rho \frac{d^2z}{ds^2};$$

a proof of this fact is supplied in Art. 54

Thus it appears that $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ *are the components along the axes of the total acceleration* j of the moving point.

54. To determine the (first) curvature $1/\rho$ and the direction cosines λ, μ, ν of the principal normal of any curve imagine the curve described by a moving point P *with constant velocity* 1. The hodograph constructed at the origin of co-ordinates, is then a spherical curve, called the *spherical indicatrix*, and the co-ordinates of the point Q of this indicatrix, corresponding to the point P of the given curve are α, β, γ. Hence, if ds' is the element of arc QQ' of the indicatrix corresponding to the arc $PP' = ds$ of the given curve, we have

$$\lambda = \frac{d\alpha}{ds'} = \frac{d}{ds'}\frac{dx}{ds} = \frac{d^2x}{ds^2}\frac{ds}{ds'}.$$

But as the radii vectores of the indicatrix are parallel to the tangents of the given curve we have (Art. 51)

$$\frac{ds'}{ds} = \frac{1}{\rho};$$

hence

$$\lambda = \rho \frac{d^2x}{ds^2},$$

and similar expressions for μ, ν.

55. When the path of P is a *plane* curve we have as components of the acceleration j along rectangular cartesian axes in the plane of motion:

$$j_x = \frac{d^2x}{dt^2} = \ddot{x}, \quad j_y = \frac{d^2y}{dt^2} = \ddot{y}.$$

When polar co-ordinates r, θ are used we may resolve the acceleration j into a component j_r along the radius vector $OP = r$ and a component j_θ at right angles to r (Fig. 15).

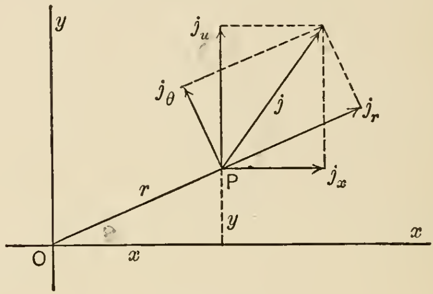

Fig. 15.

They are found by projecting $j_x = \ddot{x}$ and $j_y = \ddot{y}$ on these directions. Differentiating the relations $x = r\cos\theta$, $y = r\sin\theta$ twice with respect to t we find

$$\begin{aligned}
\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta,\\
\dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta,\\
\ddot{x} &= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta,\\
\ddot{y} &= (\ddot{r} - r\dot{\theta}^2)\sin\theta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta.
\end{aligned}$$

These expressions show directly that

$$j_r = \ddot{r} - r\dot{\theta}^2, \quad j_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}r^2\dot{\theta}.$$

56. Exercises.

(1) Show that the velocity of a moving point is increasing, constant, or diminishing according to the value of the angle ψ between v and j (Fig. 14).

(2) Show that in plane motion the sectorial velocity (Art. 47) is constant if $j_\theta = 0$, and vice versa.

(3) Show that the normal component of the acceleration is the product of the radius of curvature into the square of the angular velocity about the center of curvature.

(4) If the acceleration of a point P be always directed to a fixed point O, show that the radius vector OP describes equal areas in equal times.

(5) Show that in uniform circular motion the acceleration is directed to the center and proportional to the radius.

(6) For motion in the circle $x = a \cos\theta$, $y = a \sin\theta$ find j_x and j_y, j_r and j_θ, j_t and j_n.

(7) A wheel rolls on a straight track; find the acceleration of any point on its rim, and in particular that of its lowest and highest points.

(8) What is the hodograph (*a*) for any rectilinear motion? (*b*) for any uniform motion? (*c*) for uniform circular motion? (*d*) What can be said about the acceleration of any uniform motion?

(9) The *spherical*, or *polar*, *co-ordinates* of a point are the radius vector $r = OP$ (Fig. 16), the polar distance or colatitude $\theta = xOP$, and

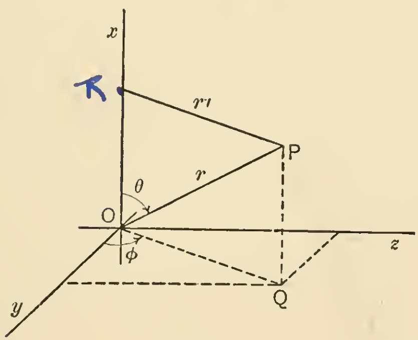

Fig. 16.

the longitude $\phi = yOQ$. The *cylindrical co-ordinates* of the same point are $r' = RP = r \sin\theta$, $\phi = yOQ$, $x = QP = r \cos\theta$. Find the cylindrical components of the acceleration (along RP, normal to xOP, and along QP), and hence show that the spherical components (along OP, perpendicular to OP in the plane xOP, and normal to xOP) are $j_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta$, $j_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{\phi}^2 \sin\theta \cos\theta$, $j_\phi = r\ddot{\phi} \sin\theta + 2\dot{r}\dot{\phi} \sin\theta + 2r\dot{\theta}\dot{\phi} \cos\theta$.

57. The fundamental problem of the kinematics of the point consists in determining the motion of the point when the acceleration is given. In cartesian co-ordinates this requires the solution of the simultaneous differential equations

$$\frac{d^2x}{dt^2} = \dot{j}_x, \quad \frac{d^2y}{dt^2} = \dot{j}_y, \quad \frac{d^2z}{dt^2} = \dot{j}_z,$$

j_x, j_y, j_z being given functions of t, x, y, z, dx/dt, dy/dt, dz/dt. A first integration would give the components of the velocity; a second integration should give the co-ordinates x, y, z as functions of the time, and hence also the path of the moving point.

It may often be more convenient to use polar co-ordinates; in the case of plane motion, we have then the equations at the end of Art. 55, with j_r and j_θ as given functions of t, r, θ and their first time-derivatives.

If the tangential and normal components of the acceleration are given we can use the equations (Art. 51):

$$\frac{dv}{dt} = \dot{j}_t, \quad \frac{v^2}{\rho} = \dot{j}_n.$$

A number of simple illustrations will be found in the following articles.

4. Examples of curvilinear motion.

(a) Constant acceleration.

58. Motion on a straight line under gravity. Let a point P move along a line inclined at the angle θ to the horizon, under the acceleration g of gravity. The motion is rectilinear; the component of the acceleration along the line is $g \sin\theta$; hence the motion is uniformly accelerated. The equations

are the same as those for falling bodies (Arts. 14, 15) except that g is replaced by $g \sin\theta$.

A particle placed on a smooth inclined plane will have this motion if its initial velocity is zero or directed along the greatest slope of the plane.

59. Exercises.

(1) Show that the final velocity is independent of the inclination; in other words, in sliding down a smooth inclined plane a body acquires the same velocity as in falling vertically through the "height" of the plane.

(2) Show that it takes a body twice as long to slide down a plane of 30° inclination as it would take it to fall through the height of the plane.

(3) At what angle θ should the rafters of a roof of given span $2b$ be inclined to make the water run off in the shortest time?

(4) Prove that the times of sliding from rest down the chords issuing from the highest (or lowest) point of a vertical circle are equal.

(5) Show how to construct geometrically the line of quickest (or slowest) descent from a given point: (a) to a given straight line, (b) to a given circle, situated in the same vertical plane.

(6) Analytically, the line of quickest or slowest descent from a given point to a curve in the same vertical plane is found by taking the equation of the curve in polar co-ordinates, $r = f(\theta)$, with the given point as origin and the axis horizontal. The time of sliding down the radius vector r is $t = \sqrt{2r/(g \sin\theta)}$. Show that this becomes a maximum or minimum when $\tan\theta = f(\theta)/f'(\theta)$, according as $f(\theta) + f''(\theta)$ is negative or positive.

(7) Show that the line of quickest descent to a parabola from its focus, the axis of the parabola being horizontal and its plane vertical, is inclined at 60° to the horizon.

60. Free motion under gravity. The motion of a point, when subject only to the constant acceleration of gravity is necessarily in the vertical plane determined by the initial velocity and the direction of gravity. Taking the horizontal line in this plane through the initial position O of the

point as axis of x, and the vertical upwards as positive axis of y (Fig. 17), the components of acceleration along

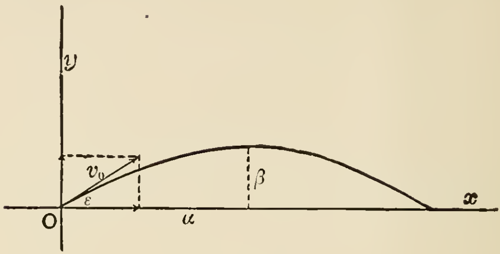

Fig. 17.

these axes are evidently 0 and $-g$, so that the equations of motion (Arts. 55, 57) are

$$\ddot{x} = 0, \quad \ddot{y} = -g.$$

The first integration gives

$$\dot{x} = c_1, \quad \dot{y} = -gt + c_2.$$

To determine the constants c_1, c_2 we must know the initial velocity in magnitude and direction. If the point starts at the time 0 from O with a velocity v_0, inclined to the horizon at an angle ϵ, the *angle of elevation*, we have for $t = 0$: $\dot{x} = v_0 \cos\epsilon$, $\dot{y} = v_0 \sin\epsilon$. Substituting these values we find $c_1 = v_0 \cos\epsilon$, $c_2 = v_0 \sin\epsilon$, so that the velocity components at any time t are:

$$\dot{x} = v_0 \cos\epsilon, \quad \dot{y} = v_0 \sin\epsilon - gt.$$

Integrating again we find

$$x = v_0 \cos\epsilon \cdot t, \quad y = \sin\epsilon \cdot t - \tfrac{1}{2}gt^2,$$

the constants of integration being 0 since $x = 0$ and $y = 0$ for $t = 0$.

These equations show that the horizontal projection of the motion is uniform, while the vertical projection is uniformly accelerated, as is otherwise apparent from the nature of the problem.

Eliminating t between the last two equations we find the *equation of the path*

$$y = \tan\epsilon \cdot x - \frac{g}{2v_0^2 \cos^2\epsilon} \cdot x^2,$$

which represents a parabola passing through the origin. To find its vertex and latus rectum, divide by the coefficient of x^2 and rearrange:

$$x^2 - \frac{2v_0^2}{g} \sin\epsilon \cos\epsilon \cdot x = -\frac{2v_0^2}{g} \cos^2\epsilon \cdot y;$$

completing the square in x, the equation can be written in the form

$$\left(x - \frac{v_0^2}{2g} \sin 2\epsilon\right)^2 = -\frac{2v_0^2}{g} \cos^2\epsilon \left(y - \frac{v_0^2}{2g} \sin^2\epsilon\right).$$

The co-ordinates of the vertex are therefore $\alpha = (v_0^2/2g)\sin 2\epsilon$, $\beta = (v_0^2/2g)\sin^2\epsilon$; the latus rectum $4a = (2v_0^2/g)\cos^2\epsilon$; the axis is vertical, and the directrix is a horizontal line at the distance $a = (v_0^2/2g) \cos^2\epsilon$ above the vertex.

61. Exercises.

(1) Show that the velocity at any time is $v = \sqrt{v_0^2 - 2gy}$.

(2) Prove that the velocity of the projectile is equal in magnitude to the velocity that it would acquire by falling from the directrix: (a) at the starting point, (b) at any point of the path (see Art. 18).

(3) Show that a body projected vertically upwards with the initial velocity v_0 would just reach the common directrix of all the parabolas described by bodies projected at different elevations ϵ with the same initial velocity v_0.

(4) The *range* of a projectile is the distance from the starting point to the point where it strikes the ground. Show that on a horizontal plane the range is $R = 2\alpha = (v_0^2/g) \sin 2\epsilon$.

(5) The *time of flight* is the whole time from the beginning of the motion to the instant when the projectile strikes the ground. It is best found by considering the horizontal motion of the projectile alone, which is uniform. Show that on a horizontal plane the time of flight is $T = (2v_0/g)\sin\epsilon$.

(6) Show that the time of flight and the range, on a plane through the starting point inclined at an angle θ to the horizon, are

$$T_\theta = \frac{2v_0}{g}\frac{\sin(\epsilon - \theta)}{\cos\theta}, \quad \text{and} \quad R_\theta = \frac{2v_0^2}{g}\frac{\sin(\epsilon - \theta)\cos\epsilon}{\cos^2\theta}.$$

(7) What elevation gives the greatest range on a horizontal plane?

(8) Show that on a plane rising at an angle θ to the horizon, to obtain the greatest range, the direction of the initial velocity should bisect the angle between the plane and the vertical.

(9) A stone is dropped from a balloon which, at a height of 625 ft., is carried along by a horizontal air-current at the rate of 15 miles an hour. (*a*) Where, (*b*) when, and (*c*) with what velocity will it reach the ground?

(10) What must be the initial velocity v_0 of a projectile if, with an elevation of 30°, it is to strike an object 100 ft. above the horizontal plane of the starting point at a horizontal distance from the latter of 1200 ft.?

(11) What must be the elevation ϵ to strike an object 100 ft. above the horizontal plane of the starting point and 5000 ft. distant, if the initial velocity be 1200 ft. per second?

(12) Show that to strike an object situated in the horizontal plane of the starting point at a distance x from the latter, the elevation must be ϵ or $90° - \epsilon$, where $\epsilon = \frac{1}{2}\sin^{-1}(gx/v_0^2)$.

(13) The initial velocity v_0 being given in magnitude and direction, show how to construct the path graphically.

(14) The solution of Ex. (11) shows that a point that can be reached with a given initial velocity can in general be reached by two different elevations. Find the locus of the points that can be reached by only one elevation, and show that it is the envelope of all the parabolas that can be described with the same initial velocity (in one vertical plane).

(15) If it be known that the path of a point is a parabola and that

the acceleration is parallel to its axis, show that the acceleration is constant.

(16) Prove that a projectile whose elevation is 60° rises three times as high as when its elevation is 30°, the magnitude of the initial velocity being the same in each case.

(17) Construct the hodograph for the motion of Art. 60, taking the focus as pole and drawing the radii vectores at right angles to the velocities.

(18) A stone slides down a roof sloping 30° to the horizon, through a distance of 12 ft. If the lower edge of the roof be 50 ft. above the ground, (*a*) when, (*b*) where, (*c*) with what velocity does the stone strike the ground?

(19) If a golf ball be driven from the tee horizontally with initial speed = 300 ft./sec., where and when would it land on ground 16 ft. below the tee if resistance of air and rotation of ball could be neglected?

(20) A man standing 15 ft. from a pole 150 ft. high aims at the top of the pole. If the bullet just misses the top where will it strike the ground if $v_0 = 1000$ ft./sec.?

62. While the type of motion discussed in Art. 60 is commonly spoken of as *projectile motion*, it should be kept in mind that it takes no account of the resistance of the air; it gives the motion of a projectile in vacuo. Owing to the very high initial velocities of modern rifle bullets, the range may be only about one tenth of what it would be according to the formulæ given above.

The study of the actual motion of a projectile in a resisting medium, such as air, forms the subject of the science of *ballistics*. See for instance **C. Cranz**, Lehrbuch der Ballistik, Vol. I, 2te Auflage, Leipzig, Teubner, 1910.

(b) The pendulum.

63. The **mathematical pendulum** is a point constrained to move in a vertical circle under the acceleration of gravity.

Let O be the center (Fig. 18), A the lowest, and B the highest point of the circle. The radius $OA = l$ of the circle is called the length of the pendulum. Any position P of the moving point is determined by the angle $AOP = \theta$

counted from the vertical radius OA in the positive (counter-clockwise) sense of rotation.

If P_0 be the initial position of the moving point at the time $t = 0$, and $\measuredangle AOP_0 = \theta_0$, then the arc $P_0P = s$ described in the time t is $s = l(\theta_0 - \theta)$; hence $v = ds/dt = -l d\theta/dt$, and $dv/dt = -l d^2\theta/dt^2$, the negative sign indicating that θ diminishes as s and t increase.

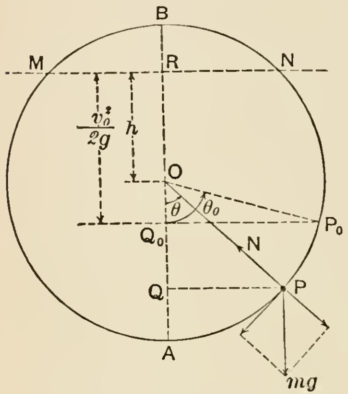

Fig. 18.

Resolving the acceleration of gravity, g, into its normal and tangential components $g \cos\theta$, $g \sin\theta$, and considering that the former is without effect owing to the condition that the point is constrained to move in a circle, we obtain the equation of motion in the form $dv/dt = g \sin\theta$, or

$$l\frac{d^2\theta}{dt^2} + g \sin\theta = 0. \tag{1}$$

64. The first integration is readily performed by multiplying the equation by $d\theta/dt$ which makes the left-hand member an exact derivative,

$$\frac{d}{dt}\left[\frac{l}{2}\left(\frac{d\theta}{dt}\right)^2 - g \cos\theta\right];$$

hence integrating, we obtain

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2 - g \cos\theta = C,$$

or considering that $v = -l d\theta/dt$,

$$\tfrac{1}{2}v^2 - gl\cos\theta = Cl.$$

To determine the constant C, the initial velocity v_0 at the time $t = 0$ must be given. We then have $\frac{1}{2}v_0^2 - gl\cos\theta_0 = Cl$; hence

$$\begin{aligned}\tfrac{1}{2}v^2 &= \tfrac{1}{2}v_0^2 - gl\cos\theta_0 + gl\cos\theta \\ &= g\left(\frac{v_0^2}{2g} - l\cos\theta_0 + l\cos\theta\right).\end{aligned} \tag{2}$$

The right-hand member can readily be interpreted geometrically; $v_0^2/2g$ is the height by falling through which the point would acquire the initial velocity v_0 (see Art. 18); $l\cos\theta - l\cos\theta_0 = OQ - OQ_0 = Q_0Q$, if Q, Q_0 are the projections of P, P_0 on the vertical AB. If we draw a horizontal line MN at the height $v_0^2/2g$ above P_0 and if this line intersect the vertical AB at R, we have for the velocity v the expression:

$$\tfrac{1}{2}v^2 = g\cdot RQ.$$

If the initial velocity be zero, the equation would be

$$\tfrac{1}{2}v^2 = g\cdot Q_0Q.$$

At the points M, N where the horizontal line MN intersects the circle the velocity becomes zero. The point can therefore never rise above these points.

Now, according to the value of the initial velocity v_0, the line MN may intersect the circle in two real points M, N, or touch it at B, or not meet it at all. In the first case the point P performs oscillations, passing from its initial position P_0 through A up to M, then falling back to A and rising to N, etc. In the third case P makes complete revolutions.

65. The second integration of the equation of motion cannot be effected in finite terms, without introducing elliptic functions. But for the case of most practical importance,

viz. for very small values of θ, it is easy to obtain an approximate solution. In this case θ can be substituted for $\sin\theta$, and the equation becomes:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0,$$

or, putting $g/l = \mu^2$:

$$\frac{d^2\theta}{dt^2} = -\mu^2\theta. \tag{3}$$

This is a well known differential equation (compare Art. 26, eq. (14), and Art. 28, Ex. 1), whose general integral is

$$\theta = C_1 \cos\mu t + C_2 \sin\mu t.$$

The constants C_1, C_2 can be determined from the initial conditions for which we shall now take $\theta = \theta_0$ and $v = 0$ when $t = 0$; this gives $C_1 = \theta_0$, $C_2 = 0$; hence

$$\theta = \theta_0 \cos\mu t,\ t = \frac{1}{\mu}\cos^{-1}\frac{\theta}{\theta_0}.$$

The last equation gives with $\theta = -\theta_0$ the time t_1 of one swing or *beat*, that is, half the period:

$$t_1 = \frac{\pi}{\mu} = \pi\sqrt{\frac{l}{g}}. \tag{4}$$

The time of a small oscillation or swing is thus seen to be independent of the arc through which the pendulum swings; in other words, for all small arcs the times of swing of the same pendulum are very nearly the same; such oscillations are therefore called *isochronous*.

66. The formula (4) shows that for a pendulum of given length l_1 the time of one swing t_1 varies for different places owing to the variation of g. As l_1 and t_1 can be measured very accurately, the pendulum can be used to determine g, the acceleration of gravity at any place; (4) gives:

$$g = \frac{\pi^2 l_1}{t_1^2}. \tag{5}$$

Now let l_0 be the length of a pendulum which *beats seconds, i. e.*, makes just one swing per second; by (4) and (5) we find for the length l_0 of such a *seconds pendulum:*

$$l_0 = \frac{g}{\pi^2} = \frac{l_1}{t_1^2}. \tag{6}$$

The length l_0 of the seconds pendulum is therefore found by measuring the length l_1 and the time of swing t_1 of any pendulum. This length l_0 is very nearly a meter; it varies slightly with g; thus, for points at the sea level it varies from $l_0 = 99.103$ cm. at the equator to $l_0 = 99.610$ at the poles.

If g_0 be the value of g at sea level, *i. e.*, at the distance R from the center of the earth, g_1 the value of g at an elevation h above sea level in the same latitude, it is known that

$$\frac{g_0}{g_1} = \frac{(R+h)^2}{R^2}.$$

Hence, if g_0 be known, pendulum experiments might serve to find the altitude of a place above sea level; but the observations would have to be of very great accuracy.

67. Let n be the number of swings made by a pendulum of length l in any time T so that $t_1 = T/n$. Then, by (4),

$$\frac{T}{n} = \pi\sqrt{\frac{l}{g}}. \tag{7}$$

If T and one of the three quantities n, l, g in this equation be regarded as constant, the small variations of the two others can be found approximately by differentiation. For instance, if the daily number of oscillations of a pendulum of constant length be observed at two different places, T and l keep the same values while n and g vary by small amounts, say Δn and Δg. Now the differentiation of (7) gives

$$-\frac{T}{n^2}dn = -\frac{\pi\sqrt{l}}{2}\frac{dg}{g^{\frac{3}{2}}},$$

or, dividing by (7):

$$\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}.$$

We have therefore approximately, for small variations Δn, Δg:

$$\frac{\Delta n}{n} = \frac{1}{2}\frac{\Delta g}{g}. \tag{8}$$

68. Exercises.

(1) Find the number of swings made in a second and in a day by a pendulum 1 meter long, at a place where $g = 980.5$.

(2) Find the length of the seconds pendulum at a place where $g = 32.17$.

(3) Find the value of g at a place where a pendulum of length 3.249 ft. is found to make 86522 swings in 24 hours.

(4) A chandelier suspended from the ceiling is seen to make 20 swings a minute; find its distance from the ceiling.

(5) A pendulum of length 1 meter is carried from the equator where $g = 978.1$ to another latitude; if it gains 100 swings a day find the value of g there.

(6) Investigate whether the approximate formula (8) is sufficiently accurate for Ex. (5).

(7) If the length of a pendulum be increased by a small amount Δl, show that the daily number of swings, n, will be diminished by Δn so that approximately

$$\frac{\Delta n}{n} = -\tfrac{1}{2}\frac{\Delta l}{l}.$$

(8) A clock beating seconds is gaining 5 minutes a day; how much should the pendulum bob be screwed up or down?

(9) A clock beating seconds at a place where $g = 32.20$ is carried to a place where $g = 32.15$; how much will it gain or lose per day if the length of the pendulum be not changed?

(10) A pendulum of length 100.18 cm. is found to beat 3585 times per hour; find the elevation of the place if in the same latitude $g = 981.02$ at sea level.

69. When the oscillations of a pendulum are not so small that the angle can be substituted for its sine, as was done in Art. 65, an expression for the time t_1 of one swing can be obtained as follows.

We have by (2), Art. 64.

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = gl(\cos\theta - \cos\theta_0).$$

Let the time be counted from the instant when the moving point has its highest position (N in Fig. 18), so that $v_0 = 0$.

Substituting $v = -l d\theta/dt$ and applying the formula $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$ we find:

$$\frac{1}{2}l\left(\frac{d\theta}{dt}\right)^2 = 2g(\sin^2\tfrac{1}{2}\theta_0 - \sin^2\tfrac{1}{2}\theta),$$

whence

$$dt = \frac{1}{2}\sqrt{\frac{l}{g}}\,\frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\theta_0 - \sin^2\frac{1}{2}\theta}}.$$

Integrating from $\theta = 0$ to $\theta = \theta_0$ and multiplying by 2 we find for the time t_1 of one swing:

$$t_1 = \sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\theta_0 - \sin^2\frac{1}{2}\theta}}.$$

As θ cannot become greater than θ_0 we may put $\sin\frac{1}{2}\theta = \sin\frac{1}{2}\theta_0 \sin\phi$, thus introducing a new variable ϕ for which the limits are 0 and $\pi/2$. Differentiating the equation of substitution, we have

$$\tfrac{1}{2}\cos\tfrac{1}{2}\theta\, d\theta = \sin\tfrac{1}{2}\theta_0 \cos\phi\, d\phi,$$

or, as $\cos\frac{1}{2}\theta = \sqrt{1 - \sin^2\frac{1}{2}\theta_0 \sin^2\phi}$,

$$d\theta = \frac{2\sin\frac{1}{2}\theta_0\cos\phi\, d\phi}{\sqrt{1 - \sin^2\frac{1}{2}\theta_0\sin^2\phi}}.$$

Substituting these values and putting for the sake of brevity

$$\sin\tfrac{1}{2}\theta_0 = \kappa,$$

we find for the time t_1 of one swing:

$$t_1 = 2\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \kappa^2\sin^2\phi}}.$$

The integral in this expression is called the complete elliptic integral of the first species and is usually denoted by K. Its value can be found from tables of elliptic integrals or by expanding the argument into an infinite series by the binomial theorem (since $\kappa\sin\phi$ is less than 1), and then performing the

integration. We have

$$(1 - \kappa^2 \sin^2\phi)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\kappa^2 \sin^2\phi + \frac{1\cdot 3}{2\cdot 4}\kappa^4 \sin^4\phi + \cdots;$$

hence

$$t_1 = \pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2\kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\kappa^4 + \cdots\right].$$

If H be the height of the initial point $N(\theta = \theta_0)$ above the lowest point A of the circle, we have

$$\kappa^2 = \sin^2\tfrac{1}{2}\theta_0 = \frac{1 - \cos\theta_0}{2} = \frac{H}{2l},$$

so that the expression for t_1 can be written in the form

$$t_1 = \pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2\frac{H}{2l} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{H}{2l}\right)^2 + \cdots\right].$$

70. Exercises.

(1) Show that $t_1 = \pi\sqrt{l/g}(1 + \frac{1}{16} + \frac{9}{1024} + \frac{25}{16384} + \cdots)$ if the angle $2\theta_0$ of the swing is 120°.

(2) Show that as second approximation to the time of a small swing we have $t_1 = \pi\sqrt{l/g}(1 + \frac{1}{16}\theta_0^2)$.

(3) Find the time of oscillation of a pendulum whose length is 1 meter at a place where $g = 980.8$, to four decimal places, the amplitude θ_0 of the swing being 6°.

(4) Denoting by t_0 the first approximation, $\pi\sqrt{l/g}$, to the time t_1 of one swing, the quotient $(t_1 - t_0)/t_0$ is called the *correction for amplitude.* Show that its value is 0.0005 for $\theta_0 = 5°$.

(5) A pendulum hanging at rest is given an initial velocity v_1. Find to what height h_1 it will rise.

(6) Discuss the pendulum problem in the particular case when MN (Fig. 18) touches the circle at B, that is when the initial velocity is due to falling from the highest point of the circle.

(c) Simple harmonic motion.

71. Simple harmonic motion is that kind of rectilinear motion in which the acceleration is proportional to the distance of the moving point P from a fixed point O in the

line of motion and is always directed toward this fixed point (Fig. 19).

An example of simple harmonic motion was discussed in Arts. 26, 27. We now resume its study from a more general point of view, owing to its great importance. It naturally

Fig. 19.

leads to the study of certain important motions known as *compound harmonic*, which may be curvilinear.

By definition, the differential equation of simple harmonic motion is

$$\ddot{x} = -\mu^2 x,$$

where μ is a constant, μ^2 being evidently the absolute value of the acceleration at the distance $x = 1$ from the origin O. The equation has the form of the pendulum equation (3), Art. 65, except that θ is replaced by x. Its general integral is therefore

$$x = C_1 \cos\mu t + C_2 \sin\mu t.$$

Differentiating, we find the velocity

$$v = -C_1\mu \sin\mu t + C_2\mu \cos\mu t.$$

If $x = x_0$ and $v = v_0$ for $t = 0$ we find $C_1 = x_0$, $C_2 = v_0/\mu$; hence

$$x = x_0 \cos\mu t + \frac{v_0}{\mu}\sin\mu t, \quad v = -x_0\mu \sin\mu t + v_0 \cos\mu t.$$

72. The expression found for x can be given a more convenient form by observing that if we construct a right-angled triangle (Fig. 20) with x_0 and v_0/μ as sides and call a its hypotenuse, ϵ its angle adjacent to x_0, we have

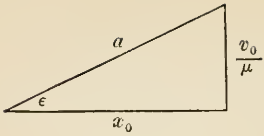

Fig. 20.

$$x_0 = a\cos\epsilon, \quad \frac{v_0}{\mu} = a\sin\epsilon;$$

substituting these values we find

$$x = a\cos\epsilon\cos\mu t + \sin\epsilon\sin\mu t$$
$$= a\cos(\mu t - \epsilon).$$

Hence, in simple harmonic motion we have

$$x = a\cos(\mu t - \epsilon), \quad v = -a\mu\sin(\mu t - \epsilon),$$

where

$$a = \sqrt{x_0^2 + \frac{v_0^2}{\mu^2}}, \quad \epsilon = \tan^{-1}\frac{v_0}{\mu x_0}.$$

The motion is clearly *periodic* since both position and velocity regain the same values when the angle $\mu t - \epsilon$ is increased by any integral multiple n of 2π, *i. e.* if the time t is increased by n times $2\pi/\mu$. The time

$$T = \frac{2\pi}{\mu}$$

between any two successive equal stages of the motion is called the **period**; the length a, which is evidently the greatest distance on either side of the origin reached by the point, is called the **amplitude** of the simple harmonic motion.

The angle $\mu t - \epsilon$ is called the **phase-angle**, ϵ the **epoch-angle** of the motion.

The point *oscillates* between the positions P_1 and P_2 (Fig. 19) whose abscissas are $\pm a$. It is at P_1 (at *elongation*) at the time $t_0 = \epsilon/\mu$ (and also at the times $t_0 + n \cdot 2\pi/\mu = (\epsilon + 2n\pi)/\mu$); it reaches the position O at the time $t_1 = (\epsilon + \frac{1}{2}\pi)/\mu$, so that the time of passing from P_1 to O is

$$t_1 - t_0 = \frac{\pi}{2\mu}.$$

The time of passing from O to the other elongation P_2 is

easily shown to be equal to this; so that the time of one swing (from P_1 to P_2) is

$$\frac{\pi}{\mu}.$$

The backward motion from P_2 to P_1 takes place in the same time so that the *period*, that is the time of a double (forward and backward) swing, is, as shown above,

$$T = \frac{2\pi}{\mu}.$$

73. An instructive illustration is obtained by observing that *any simple harmonic motion can be regarded as the projection of a uniform circular motion on a diameter of the circle.* In other words, it is the *apparent* motion of a point describing a circle uniformly, as seen from a point in the plane of the circle (at an infinite distance). For, let a point Q (Fig. 21) describe a circle of radius a with constant angular

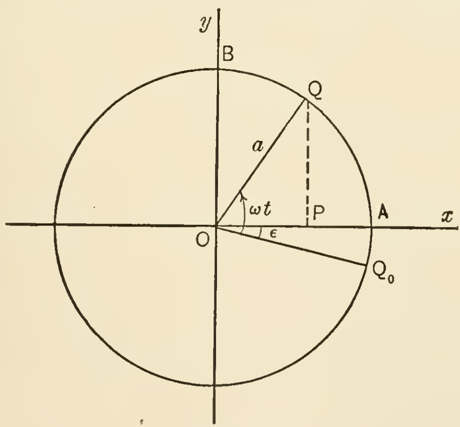

Fig. 21.

velocity ω, say in the counterclockwise sense. If Q_0 is the position of the point at the time $t = 0$, we have $Q_0OQ = \omega t$, so that the projection of Q on the diameter OQ_0 has, for the center O as origin, the abscissa

$a \cos\omega t$. And if P be the projection of Q on a diameter OA making with OQ_0 the angle ϵ, the abscissa of P will be

$$x = a \cos(\omega t - \epsilon).$$

Hence the motion of P is a simple harmonic motion for which the acceleration at unit distance from O is $\mu^2 = \omega^2$.

74. Notice that the linear velocity $v = a\omega$ of Q has along OA the component

$$v_x = \dot{x} = - a\omega \sin(\omega t - \epsilon),$$

which is the velocity of P; and the acceleration of Q, $j = a\omega^2$ along QO, has along OA the component

$$j_x = \ddot{x} = - a\omega^2 \cos(\omega t - \epsilon) = - \omega^2 x,$$

which is the acceleration of P.

The projection of the uniform circular motion of Q on the diameter OB, perpendicular to OA, gives also a simple harmonic motion, viz.

$$y = a \sin(\omega t - \epsilon) = a \cos[\omega t - (\epsilon + \tfrac{1}{2}\pi)],$$

which merely differs by $\frac{1}{2}\pi$ in phase from the motion along OA.

The period of the simple harmonic motion of P along OA is (Art. 72):

$$T = 2\pi/\omega,$$

i. e., it is equal to the time in which Q makes one revolution on the circle. The fact that this period depends only on the angular velocity and not on the radius a, *i. e.* on the amplitude, is expressed by saying that simple harmonic motions of the same μ or ω are **isochronous.**

If Q describes the circle p times per second so that P makes p complete (forth and back) oscillations per second, we have $\omega = 2\pi p$, so that

$$T = 1/p;$$

i. e. the number of oscillations per second, the so-called **frequency,** *is the reciprocal of the period.*

75. Exercises.

(1) Integrate the equation $\ddot{x} = - \mu^2 x$, by multiplying it by $\dot{x}$, and determine the constants of integration if $x = x_0$, $v = v_0$ for $t = 0$.

(2) Show that the period T can be expressed in the form $2\pi\sqrt{- x/\ddot{x}}$; also find the velocity in terms of x.

(d) Compound harmonic motion.

76. Apart from the initial conditions, a simple harmonic motion is fully determined by its line l, its center O, and its period (or frequency), which determines the constant μ. The amplitude a and the phase ϵ depend on the initial conditions (see Art. 72).

Let a point P have a simple harmonic motion of period $T = 2\pi/\mu$ along a line l, about the center O; and let the line l have a motion of rectilinear translation in a fixed plane π (comp. Art. 38). If the motion of l is likewise a simple harmonic motion, about O as center, in a direction l', the absolute motion of P in the plane π is called a **compound harmonic motion.** This is in general a curvilinear motion; but it becomes rectilinear when the direction l' is parallel to l.

We proceed to examine in some detail the most important cases of this **composition of** two or more **simple harmonic motions,** beginning with those cases in which the resultant motion is rectilinear.

As, according to Hooke's law, the particles of elastic bodies, after release from strain within the elastic limits, perform small oscillations for which the acceleration is proportional to the displacement from a middle position, the motions under discussion find a wide application in the theories of elasticity, sound, light, and electricity, and form the basis of the general theory of wave motion in an elastic medium.

77. *Two simple harmonic motions in the same line, of equal period T, but differing in amplitude and phase, compound into a single simple harmonic motion in the same line and of the same period.*

For, by Art. 72, the component displacements can be written

$$x_1 = a_1 \cos(\omega t + \epsilon_1), \quad x_2 = a_2 \cos(\omega t + \epsilon_2),$$

and being in the same line they can be added algebraically, giving the resultant displacement

$$\begin{aligned} x = x_1 + x_2 &= a_1 \cos(\omega t + \epsilon_1) + a_2 \cos(\omega t + \epsilon_2) \\ &= (a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2) \cos\omega t - (a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2) \sin\omega t. \end{aligned}$$

Putting (comp. Art. 72)

$$a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2 = a \cos\epsilon, \quad a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2 = a \sin\epsilon,$$

we have

$$x = a \cos\epsilon \cos\omega t - \sin\epsilon \sin\omega t = a \cos(\omega t + \epsilon),$$

where

$$\begin{aligned} a^2 &= (a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2)^2 + (a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2)^2 \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\epsilon_2 - \epsilon_1) \end{aligned}$$

and

$$\tan\epsilon = \frac{a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2}{a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2}.$$

78. A geometrical illustration of the preceding proposition is obtained by considering the uniform circular motions corresponding to the two simple harmonic motions (Fig. 22).

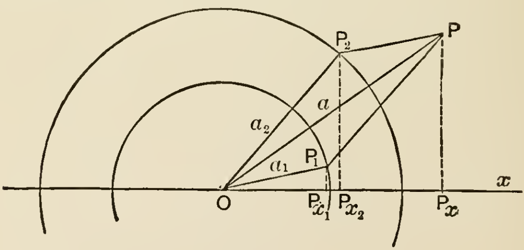

Fig. 22.

Drawing the radii $OP_1 = a_1$, $OP_2 = a_2$ so as to include an angle equal to the difference of phase $\epsilon_2 - \epsilon_1$ and completing the parallelogram OP_1PP_2, it appears from the figure that the diagonal OP of this parallelogram represents the resulting amplitude a.

As P_1P is equal and parallel to OP_2, we have for the projections on any axis Ox the relation $OP_{x_1} + OP_{x_2} = OP$, or $x_1 + x_2 = x$. If now the axis Ox be drawn so as to make the angle xOP_1 equal to the epoch-angle ϵ_1, and hence $xOP_2 = \epsilon_2$, the angle xOP represents the epoch ϵ of the resulting motion.

We thus have a simple geometrical construction for the elements a, ϵ of the resulting motion from the elements a_1, ϵ_1 and a_2, ϵ_2 of the component motions. As the period is the same for the two component motions, the points P_1 and P_2 describe their respective circles with equal angular velocity so that the parallelogram OP_1PP_2 does not change its form in the course of the motion.

79. The construction given in the preceding article can be described briefly by saying that two simple harmonic motions of equal period in the same line are compounded by *geometrically adding* their amplitudes, it being understood that the phase-angles determine the directions in which the amplitudes are to be drawn. Analytically, this appears of course directly from the formulæ of Art. 77.

It follows at once that not only two, but *any number of simple harmonic motions, of equal period in the same line, can be compounded by geometric addition of their amplitudes into a single simple harmonic motion in the same line and of the same period.*

Conversely, any given simple harmonic motion can be resolved into two or more components in the same line and of the same period.

80. Exercises.

(1) Find the resultant of three simple harmonic motions in the same line, and all of period $T = 12$ seconds, the amplitudes being 5, 3, and 4 cm., and the phase differences 30° and 60°, respectively, between the first and second, and the first and third motions.

(2) If in the proposition of Art. 77 the amplitudes are equal, $a_1 = a_2 = a$, while the phase-angles differ by $\epsilon_2 - \epsilon_1 = \delta$, show that the resulting motion has the amplitude $2a \cos\frac{1}{2}\delta$ and the phase-angle $\frac{1}{2}\delta$: (*a*) directly, (*b*) from the formulæ of Art. 77, (*c*) by the geometric method of Art. 78.

(3) Find the resultant of two simple harmonic motions in the same line and of equal period when the amplitudes are equal and the phases differ: (*a*) by an even multiple of π, (*b*) by an odd multiple of π.

(4) Resolve $x = 10 \cos(\omega t + 45°)$ into two components in the same

line with a phase difference of 30°, one of the components having the epoch 0.

(5) Trace the curves representing the component motions as well as the resultant motion in Ex. (1), taking the time as abscissa and the displacement as ordinate.

(6) Show that the resultant of n simple harmonic motions of equal period T in the same line, viz.

$$x_i = a_i \cos\left(\frac{2\pi}{T}t + \epsilon_i\right),$$

is the isochronous simple harmonic motion

$$x = a \cos\left(\frac{2\pi}{T}t + \epsilon\right),$$

where

$$a^2 = \left(\sum_1^n a_i \cos\epsilon_i\right)^2 + \left(\sum_1^n a_i \sin\epsilon_i\right)^2, \quad \tan\epsilon = \frac{\sum_1^n a_i \sin\epsilon_i}{\sum_1^n a_i \cos\epsilon_i}.$$

81. The composition of two or more simple harmonic motions in the same line can readily be effected, even *when the components differ in period. But the resultant motion is in general not simple harmonic.*

Thus, with two components

$$x_1 = a_1 \cos(\omega_1 t + \epsilon_1), \quad x_2 = a_2 \cos(\omega_2 t + \epsilon_2),$$

putting $\omega_2 t + \epsilon_2 = \omega_1 t + (\omega_2 - \omega_1)t + \epsilon_2 = \omega_1 t + \epsilon_1 + \delta$, say, where $\delta = (\omega_2 - \omega_1)t + \epsilon_2 - \epsilon_1$ is the difference of phase at the time t, we have for the resulting motion

$$x = x_1 + x_2 = a_1 \cos(\omega_1 t + \epsilon_1) + a_2 \cos(\omega_1 t + \epsilon_1 + \delta);$$
$$= (a_1 + a_2 \cos\delta) \cos(\omega_1 t + \epsilon_1) - a_2 \sin\delta \sin(\omega_1 t + \epsilon_1),$$

or putting $a_1 + a_2 \cos\delta = a \cos\epsilon, \quad a_2 \sin\delta = a \sin\epsilon$:

$$x = a \cos(\omega_1 t + \epsilon_1 + \epsilon),$$

where

$$a^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta, \quad \tan\epsilon = \frac{a_2 \sin\delta}{a_1 + a_2 \cos\delta},$$
$$\delta = (\omega_2 - \omega_1)t + \epsilon_2 - \epsilon_1.$$

It can be shown that this represents a simple harmonic motion only when $\omega_2 = \pm \omega_1$.

The formulæ can be interpreted geometrically by Fig. 22 as in Art. 78. But as in the present case the angle δ, and consequently the quantities a and ϵ in the expression for x, vary with the time, the parallelogram OP_1PP_2 while having constant sides has variable angles and changes its form in the course of the motion.

(e) Wave motion.

82. To show the connection of the present subject with the theory of **wave motion**, imagine a flexible cord AB of which one end B is fixed, while the other A is given a sudden

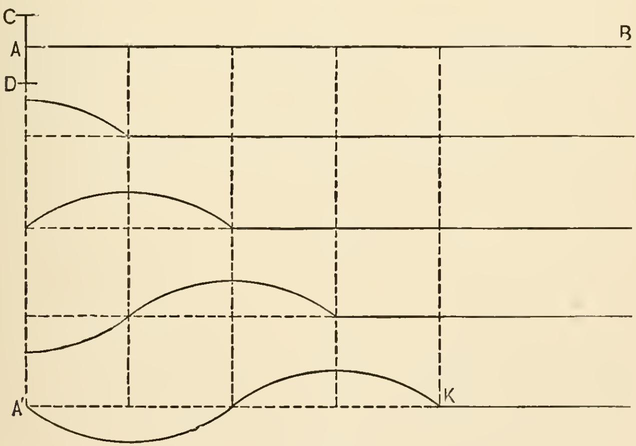

Fig. 23.

jerk or transverse motion from A to C and back through A to D, etc. (Fig. 23). The displacement given to A will, so to speak, run along the cord, travelling from A to B and producing a wave, while any particular point of the cord

has approximately a rectilinear motion at right angles to AB. The figure exhibits the successive stages of the motion up to the time when a complete wave $A'K$ has been produced.

The distance $A'K = \lambda$ is called the **length of the wave.** Let T be the time in which the motion spreads from A' to K, that is, the time of a complete vibration of the point A, from A to C, back to D, and back again to A; then

$$\frac{\lambda}{T} = V$$

is called the **velocity of propagation** of the wave.

83. Suppose now that the vibration of A is a simple harmonic motion, say $y = a \sin \omega t$. As the time of vibration of A is T we must have $\omega = 2\pi/T$, and hence

$$\omega = \frac{2\pi}{\lambda} V.$$

If we assume that the vibrations of the successive points of the cord differ from the motion of A only in phase, the displacements of all points of the cord at any time t can be represented by

$$y = a \sin(\omega t - \epsilon),$$

where ϵ varies from 0 to 2π as we pass from A' to K.

If we further assume that the phase-angle ϵ of any point of the cord is proportional to the distance x of the point from A' we have $\epsilon = kx$, or since $\epsilon = 2\pi$ for $x = \lambda$:

$$\epsilon = \frac{2\pi}{\lambda} x.$$

Substituting the values of ω and ϵ we find

$$y = a \sin \left[\frac{2\pi}{\lambda} (Vt - x) \right]. \tag{9}$$

The assumptions here made can be regarded as roughly

suggested by the experiment of Art. 82 or similar observations. The motion represented by the final equation (9) may be called **simple harmonic wave motion.**

84. To understand the full meaning of the equation (9) it should be observed that, as (in accordance with the assumptions of Art. 83) the quantities a, λ, V are regarded as constant, the displacement y is a function of the two variables t and x.

If t be given a particular value t_1, equation (9) represents the displacements of all points of the cord at the time t_1. The substitution for x of $x + n\lambda$, where n is any positive or negative integer, changes the angle $(2\pi/\lambda)$ $(Vt - x)$ by $2\pi n$ and hence leaves y unchanged. This means that the displacements of all points whose distances from A differ by whole wave-lengths are the same; in other words, the state of motion at any instant is given by a series of equal waves.

If, on the other hand, we assign a particular value x_1 to x and let t vary, the equation represents the rectilinear vibration of the point whose abscissa is x_1. By substituting for t the value $t + nT = t + n\lambda/V$, the angle $(2\pi/\lambda)(Vt - x)$ is again changed by $2\pi n$, so that y remains unchanged. This shows the periodicity of the motion of any point.

85. It may be well to state once more, and as briefly as possible, the fundamental assumptions that underlie the important formula (9).

The idea of simple harmonic wave motion implies that the displacement y should be a periodic function of x and t such as to fulfil the following conditions: y must assume the same value (a) when x is changed to $x + n\lambda$, (b) when t is changed to $t + nT$, (c) when both changes are made simultaneously; the constants λ and T being connected by the relation $\lambda = VT$.

The condition (c) requires y to be of the form $y = f(Vt - x)$; for $Vt - x$ remains unchanged when x is replaced by $x + n\lambda$ and at the same time t by $t + nT$.

6

A particular case of such a function is $y = a \sin c(Vt - x)$. As y should remain unchanged when t is replaced by $t + T$, we must have $c = 2\pi/VT = 2\pi/\lambda$. Thus the function

$$y = a \sin \frac{2\pi}{\lambda} (Vt - x)$$

fulfils the three conditions (*a*), (*b*), (*c*).

Putting $2\pi x/\lambda = -\epsilon$ we have

$$y = a \sin \left(\frac{2\pi}{T} t + \epsilon \right).$$

The importance of this particular solution of our problem lies in the fact that, according to *Fourier's theorem*, any single-valued periodic function of period T can be expanded, between definite limits of the variable, in a series of the form:

$$f(t) = a_0 + a_1 \sin \left(\frac{2\pi}{T} \cdot t + \epsilon_1 \right) + a_2 \sin \left(\frac{2\pi}{T} \cdot 2t + \epsilon_2 \right)$$
$$+ a_3 \sin \left(\frac{2\pi}{T} \cdot 3t + \epsilon_3 \right) + \cdots.$$

As applied to the theory of wave motion this means that any wave motion, however complex, can be regarded as made up of a series of superposed simple harmonic wave motions of periods T, $\frac{1}{2}T$, $\frac{1}{3}T$, . . ., or since $T = \lambda/V$, of wave-lengths λ, $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, ... For, if the point A (Fig. 23) be subjected simultaneously to more than one simple harmonic motion, the displacements resulting from each can be added algebraically, thus forming a compound wave which can readily be traced by first tracing the component waves and then adding their ordinates.

The motion due to the superposition of two or more simple harmonic waves may be called *compound harmonic wave motion.*

86. Exercises.

(1) Trace the wave produced by the superposition of two simple harmonic wave motions in the same line of equal amplitudes, the periods being as 2 : 1, (*a*) when they do not differ in phase, (*b*) when their epochs differ by 7/16 of the period.

(2) In the problem of Art. 81, determine the maximum and minimum of the resulting amplitude a and show that the number of maxima

per second is equal to the difference of the number of vibrations per second.

(f) Curvilinear compound harmonic motion.

87. An important and typical case is the motion of a point P whose acceleration j is directed toward a fixed center O

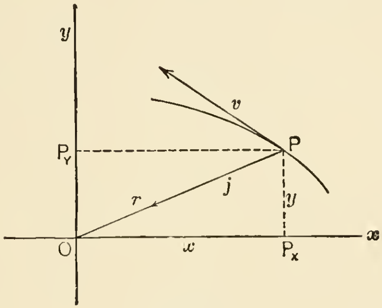

Fig. 24.

and proportional to the distance $OP = r$ from this center (Fig. 24).

If the initial velocity is $\neq 0$ and does not happen to pass through the center O, the motion is curvilinear. But it is confined to the plane determined by the center and the initial velocity since the acceleration $j = \mu^2 r$ lies in this plane.

Taking the center O as origin and any rectangular axes Ox, Oy in this plane, we have for the direction cosines of OP: x/r, y/r, for those of the acceleration: $-x/r$, $-y/r$, so that the equations of motion are

$$\ddot{x} = -\mu^2 x, \quad \ddot{y} = -\mu^2 y.$$

These equations show that the projections P_x, P_y of P on the axes have each a simple harmonic motion, of the same center and period. The motion of P is the absolute motion of a point having a simple harmonic motion of period $2\pi/\mu$

along the axis Ox, about O, while this axis itself has a simple harmonic motion of the same period about O along the axis Oy.

Each of the two equations is readily integrated, and by eliminating t it is found that the path is an ellipse, with O as center. See Arts. 298–302.

88. To *compound any number of simple harmonic motions not in the same line* observe that the projection of a simple harmonic motion on any line is again a simple harmonic motion of the same period and phase and with an amplitude equal to the projection of the original amplitude.

For the sake of simplicity we confine ourselves to the case of motions in the same plane and with the same center O. Projecting all the simple harmonic motions on two rectangular axes Ox, Oy, we can, by Arts. 77, 79, compound the components in each axis; it then only remains to find the resultant of the two motions along Ox and Oy.

89. Just as in Arts. 77, 81, we must distinguish two cases: (*a*) When the given motions have all the same period, and (*b*) when they have not.

In the former case, by Art. 77, the two components along Ox and Oy will have equal periods, *i. e.* they will be of the form

$$x = a\cos\omega t, \quad y = b\cos(\omega t + \delta).$$

The path of the resulting motion is obtained by eliminating t between these equations. We have

$$\frac{y}{b} = \cos\omega t\cos\delta - \sin\omega t\sin\delta$$

$$= \frac{x}{a}\cos\delta - \sqrt{1 - \frac{x^2}{a^2}}\sin\delta.$$

Writing this equation in the form

$$\left(\frac{y}{b} - \frac{x}{a}\cos\delta\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta,$$

or

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\delta + \frac{y^2}{b^2} = \sin^2\delta, \tag{10}$$

we see that it represents an ellipse (since $\frac{1}{a^2}\cdot\frac{1}{b^2} - \frac{\cos^2\delta}{a^2b^2} = \left(\frac{\sin\delta}{ab}\right)^2$

is positive) whose center is at the origin. The resultant motion is therefore called **elliptic harmonic motion.**

We have thus the general result that *any number of simple harmonic motions of the same period and in the same plane, whatever may be their directions, amplitudes, and phases, compound into a single elliptic harmonic motion.*

90. A few particular cases may be noticed. The equation (10) will represent a (double) straight line, and hence the elliptic vibration will degenerate into a simple harmonic vibration, whenever $\sin^2\delta = 0$, *i. e.* when $\delta = n\pi$, where n is a positive or negative integer. In this case $\cos\delta$ is $+ 1$ or -1, and (10) reduces to

$$\frac{x}{a} - \frac{y}{b} = 0, \text{ if } \delta = 2m\pi,$$

and to

$$\frac{x}{a} + \frac{y}{b} = 0, \text{ if } \delta = (2m + 1)\pi.$$

Thus two rectangular vibrations of the same period compound into a simple harmonic vibration when they differ in phase by an integral multiple of π, that is when one lags behind the other by half a wave-length.

Again, the ellipse (10) reduces to a circle only when $\cos\delta = 0$, *i. e.* $\delta = (2m + 1)\pi/2$, and in addition $a = b$, the co-ordinates being assumed orthogonal.

Thus two rectangular vibrations of equal period and amplitude compound into a circular vibration if they differ in phase by $\pi/2$, *i. e.* if one lags behind the other by a quarter of a wave-length.

This circular harmonic motion is evidently nothing but uniform motion in a circle; and we have seen in Art. 73 that, conversely, uniform circular motion can be resolved into two rectangular simple harmonic vibrations of equal period and amplitude, but differing in phase by $\pi/2$.

91. It remains to consider the case when the given simple harmonic motions do not all have the same period. It follows from Art. 81 that in this case, if we again project the given motions on two rectangular axes Ox, Oy, the resulting motions along Ox, Oy are in general not simple harmonic.

The elimination of t between the expressions for x and y may present difficulties. But, of course, the curve can always be traced by points, graphically.

We shall here consider only the case when the motions along Ox and Oy are simple harmonic.

92. *If two simple harmonic motions along the rectangular directions* Ox, Oy, viz.:

$$x = a_1 \cos(\omega_1 t + \epsilon_1), \quad y = a_2 \cos(\omega_2 t + \epsilon_2),$$

of different amplitudes, phases, and *periods* are to be compounded, the resulting motion will be confined within a rectangle whose sides are $2a_1$, $2a_2$, since these are the maximum values of $2x$ and $2y$.

The path of the moving point will be a *closed* curve only when the quotient $\omega_2/\omega_1 = T_1/T_2$ is a rational number, say $= m/n$, where m is prime to n. The x co-ordinate of the curve will have m maxima, the y co-ordinate n, and the whole curve will be traversed after m vibrations along Ox and n along Oy.

The formation of the resulting curve will best be understood from the following example.

93. Let $a_1 = a_2 = a$, $\epsilon_1 = 0$, $\epsilon_2 = \delta$, and let the ratio of the periods be $T_1/T_2 = 2/1$. The equations of the component simple harmonic vibrations are

$$x = a \cos\omega t, \quad y = a \cos(2\omega t + \delta).$$

Here it is easy to eliminate t. We have

$$y = a \cos 2\omega t \cos\delta - a \sin 2\omega t \sin\delta$$

$$= a\left(2\frac{x^2}{a^2} - 1\right)\cos\delta - 2a\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\sin\delta.$$

Hence the equation of the path is:

$$ay = (2x^2 - a^2)\cos\delta - 2x\sqrt{a^2 - x^2}\sin\delta.$$

If there be no difference of phase between the components, *i. e.* if $\delta = 0$, this reduces to the equation of a parabola:

$$x^2 = \tfrac{1}{2}a(y + a).$$

For $\delta = \pi/2$, the equation also assumes a simple form:

$$a^2y^2 = 4x^2(a^2 - x^2).$$

94. It is instructive to trace the resulting curves for a given ratio of periods and for a series of successive differences of phase (*Lissajous's Curves*).

Thus in Fig. 25, the curve for $T_1/T_2 = 3/4$, and for a phase difference $\delta = 0$ is the heavily drawn curve, while the dotted curve repre-

sents the path for the same ratio of the periods when the phase difference is one-twelfth of the smaller period. The equations of the components are for the heavy curve

$$x = 6 \cos \frac{2\pi}{3} t, \quad y = 5 \cos \frac{2\pi}{4} t,$$

and for the dotted curve

$$x = 6 \cos \left(\frac{2\pi}{3} t + \frac{2\pi}{12} \right), \quad y = 5 \cos \frac{2\pi}{4} t.$$

In tracing these curves, imagine the simple harmonic motions replaced by the corresponding uniform circular motions (Fig. 25).

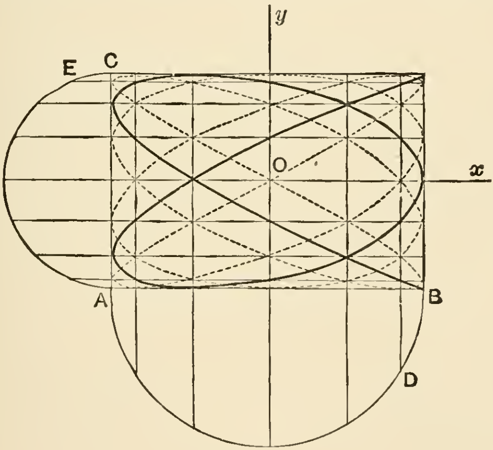

Fig. 25.

With the amplitudes 6, 5, as radii, describe the semi-circles ADB, AEC, so that BC is the rectangle within which the curves are confined; the intersection of the diagonals of this rectangle is the origin O, AB is parallel to the axis of x, AC to the axis of y. Next divide the circles on AB, AC into parts corresponding to equal intervals of time. In the present case, the periods for AB, AC being as 3 to 4, the circle on AB must be divided into $3n$ equal parts, that on AC into $4n$. In the figure, n is taken as 4, the circles being divided into 12 and 16 equal parts, respectively.

The first point of the heavily drawn curve corresponds to $t = 0$, that is $x = 6$, $y = 5$; this gives the upper right-hand corner of the rectangle. The next point is the intersection of the vertical line through D and the horizontal line through E, the arcs $BD = 1/12$ of the circle over AB, and $CE = 1/16$ of that over AC being described in the same time, so that the co-ordinates of the corresponding point are

$$x = 6 \cos\left(\frac{2\pi}{3}\cdot\frac{3}{12}\right) = 6 \cos\left(2\pi\cdot\frac{1}{12}\right),$$

$$y = 5 \cos\left(\frac{2\pi}{4}\cdot\frac{4}{16}\right) = 5 \cos\left(2\pi\cdot\frac{1}{16}\right).$$

Similarly the next point

$$x = 6 \cos\left(2\pi\cdot\frac{2}{12}\right),\quad y = 5 \cos\left(2\pi\cdot\frac{2}{16}\right)$$

is found from the next two points of division on the circles, etc.

To construct the dotted curve, it is only necessary to begin on the circle over AB with D as first point of division.

95. Exercises.

(1) With the data of Art. 94 construct the curves for phase differences of 2/12, 3/12, . . . 11/12 of the smaller period.

(2) Construct the curves (Art. 93)

$$x = a \cos\omega t,\quad y = a \cos(2\omega t + \delta)$$

for $\delta = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4, 2\pi$.

(3) Trace the path of a point subjected to two circular vibrations of the same amplitude, but differing in period: (*a*) when the sense is the same; (*b*) when it is opposite.

(g) Central motion.

96. The motion of a point P is called **central** if the *direction* of the acceleration always passes through a fixed point O. It will here be assumed, in addition, that the *magnitude* of the acceleration is a function of the distance $OP = r$ alone, say

$$j = f(r).$$

The fixed point O is in this case usually regarded as the seat of an attractive or repulsive force producing the acceleration, and is therefore called the *center of force*.

Harmonic motion as discussed in Art. 71 sq. is a special case of central motion, viz. the case in which the acceleration j is directly proportional to the distance from the fixed center O, *i. e.* $f(r) = \mu r$; see Art. 87.

Another very important particular case is that of Newton's law, *i. e.* $f(r) = \mu/r^2$; this will be discussed below, Arts. 109–112.

97. Any central motion is fully determined if in addition to the form of the function $f(r)$ we know the "initial conditions," say the initial distance $OP_0 = r_0$ (Fig. 26) and the

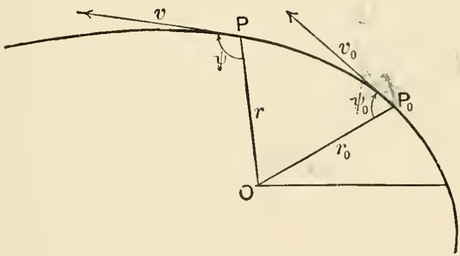

Fig. 26.

initial velocity v_0 of the point at the time $t = 0$. As v_0 must be given both in magnitude and direction, the angle ψ_0 between r_0 and v_0 must be known.

It is evident, geometrically, that the motion is confined to the plane determined by O and v_0 since the acceleration always lies in this plane. Hence, *any central motion*, whatever may be the law of acceleration, *is a plane motion.*

98. Another fundamental property is that *in any central motion*, whatever the law of acceleration, *the sectorial velocity is constant.* This is most readily proved by taking the center O as origin for polar co-ordinates r, θ. As by the definition of central motion (Art. 96) the acceleration j is directed

along the radius vector $OP = r$ drawn from the center O to the moving point P, the component j_θ of the acceleration, at right angles to the radius vector, is always zero. We have therefore by the last of the equations of Art. 55:

$$j_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0,$$

whence

$$r^2 \frac{d\theta}{dt} = c, \tag{11}$$

where c is the constant of integration. By Art. 47 this equation means that the sectorial velocity is constant and equal to $\frac{1}{2}c$.

99. Let S be the sector P_0OP described by the radius vector r in the time t, so that $dS = \frac{1}{2}r^2 d\theta$ is the elementary sector described in the element of time dt. Then (11) can be written

$$\frac{dS}{dt} = \frac{1}{2}c,$$

whence integrating, since $S = 0$ for $t = 0$:

$$S = \tfrac{1}{2}ct.$$

This shows that *the sector is proportional to the time in which it is described*, which is merely another way of stating that the sectorial velocity is constant.

It can be shown conversely, by reversing the steps of the above argument, that if in a plane motion the areas swept out by the radius vector drawn from a fixed point of the plane are proportional to the time, the acceleration must constantly pass through that point

It is well known that Kepler had found by a careful examination of the observations available to him that *the orbits described by the planets are plane curves, and the sector described*

by the radius vector drawn from the sun to any planet is proportional to the time in which it is described. This constitutes **Kepler's first law** of planetary motion.

He concluded from it that the acceleration must constantly pass through the sun.

100. To express the value of the constant of integration c in terms of the given initial conditions (Art. 97), *i. e.* by means of r_0, v_0, ψ_0, observe that at any time t

$$c = r^2\frac{d\theta}{dt} = r\cdot\frac{rd\theta}{ds}\cdot\frac{ds}{dt} = r\sin\psi\cdot v;$$

hence at the time $t = 0$ we have

$$c = v_0 r_0 \sin\psi_0.$$

Denoting by p_0 and p the perpendiculars let fall from O on v_0 and v we have $r_0\sin\psi_0 = p_0$, $r\sin\psi = p$; hence

$$c = p_0 v_0 = pv,$$

i. e. the velocity at any time is inversely proportional to its distance from the center.

101. Let us now assume that the acceleration j of a central motion is a given function, $f(r)$, of the radius vector $OP = r$ drawn from the center O to the moving point P. With O as origin, let x, y be the rectangular cartesian co-ordinates of the moving point P, and r, θ its polar co-ordinates, at any time t. Then $\cos\theta = x/r$, $\sin\theta = y/r$ are the direction cosines of $OP = r$, and, therefore, those of the acceleration j, provided the sense of j be away from the center, *i. e.* provided the force causing the acceleration be *repulsive*. In the case of *attraction*, the direction cosines of j are of course $-x/r$, $-y/r$.

Thus the *equations of motion* are in the case of attraction:

$$\ddot{x} = -f(r)\frac{x}{r}, \quad \ddot{y} = -f(r)\frac{y}{r}. \tag{12}$$

For repulsion, it would only be necessary to change the sign of $f(r)$.

To integrate the equations (12) we cannot, in general, treat each equation by itself; for, as $r = \sqrt{x^2 + y^2}$, each equation contains three variables x, y, t. We must therefore try to combine the equations so as to form integrable combinations.

102. Let us first multiply the equations (12) by y, x and subtract; the right-hand member of the resulting equation is zero, while the left-hand member is an exact derivative:

$$x\ddot{y} - y\ddot{x} \equiv \frac{d}{dt}(x\dot{y} - y\dot{x}) = 0.$$

Integrating we find $x\dot{y} - y\dot{x} = c$, or in polar co-ordinates

$$r^2\dot{\theta} = c,$$

which is the equation (11) of Art. 98.

103. Next multiply the equations (12) by $\dot{x}$, $\dot{y}$ and add; the left-hand member of the resulting equation is

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \frac{d}{dt}(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2) = \frac{d}{dt}\tfrac{1}{2}v^2;$$

the right-hand member becomes

$$-\frac{f(r)}{r}(x\dot{x} + y\dot{y}) = -\frac{f(r)}{r}\frac{d}{dt}\tfrac{1}{2}(x^2 + y^2) =$$

$$-\frac{f(r)}{r}\frac{d}{dt}(\tfrac{1}{2}r^2) = -f(r)\dot{r}.$$

The resulting equation

$$d\tfrac{1}{2}v^2 = -f(r)dr \tag{13}$$

gives

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = -\int_{r_0}^{r} f(r)dr; \tag{14}$$

i. e. it determines the *velocity as a function of r*.

104. The two methods of combining the differential equations of motion (12) used in Arts. 102 and 103 are known, respectively, as the *principle of areas* and the *principle of kinetic energy and work.* The former name explains itself (see Arts. 98, 99). The latter is due to the fact (to be more fully explained in kinetics) that if equation (14) be multiplied by the mass of the moving body, the left-hand member will represent the increase in kinetic energy while the right-hand member is the work of the central force.

Each of these methods of preparing the equations of motion for integration consists merely in combining the equations so as to obtain an exact derivative in the left-hand member of the resulting equation. If by this combination the right-hand member happens to vanish or to become likewise an exact derivative, an integration can at once be performed. This is the case in our problem.

105. The two equations (11) and (14), each of which was found by a first integration, are called *first integrals* of the equations of motion. By combining them and integrating again, the equation of the path is found.

We have, by the last equation of Art. 46, for *any* curvilinear motion

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right];$$

eliminating $d\theta/dt$ by means of (11) we find for any central motion:

$$v^2 = \frac{c^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] = c^2\left[\left(\frac{d}{d\theta}\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2\right]. \qquad (15)$$

Substituting this expression of v^2 in (14) we have the differential equation of the path in which the variables are separable. Shorter methods may occasionally suggest themselves in particular cases; see, for instance, Art. 110.

106. To solve the converse problem, viz. to find the law of acceleration when the path is known, we have only to substitute the expression (15) of v^2 in the equation (13).

In doing this it is found convenient to introduce instead of r its reciprocal $u = 1/r$, so that

$$v^2 = c^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right],$$

and to change the r-derivative of $\frac{1}{2}v^2$ to a θ-derivative since r, and hence u, is now a given function of θ. As $du/dr = -1/r^2 = -u^2$, we find

$$\begin{aligned} f(r) &= -\frac{d}{dr}\tfrac{1}{2}v^2 = -\frac{d\frac{1}{2}v^2}{d\theta}\frac{d\theta}{dr} = -\frac{d\frac{1}{2}v^2}{d\theta}\frac{d\theta}{du}\frac{du}{dr} \\ &= u^2\frac{d\theta}{du}\frac{d}{d\theta}\tfrac{1}{2}v^2 = c^2u^2\frac{d\theta}{du}\left(\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + u\frac{du}{d\theta}\right); \end{aligned}$$

hence

$$f(r) = c^2u^2\left(\frac{d^2u}{d\theta^2} + u\right). \tag{16}$$

This important relation can also be obtained directly from the equations of motion in polar co-ordinates which are (see Art. 55)

$$\ddot{r} - r\dot{\theta}^2 = -f(r), \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0.$$

For, with $r = 1/u$ we have since the second equation gives (11):

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{c}{r^2}\frac{dr}{d\theta} = -c\frac{du}{d\theta}, \quad \ddot{r} = -c\frac{d^2u}{d\theta^2}\dot{\theta} = -c^2u^2\frac{d^2u}{d\theta^2};$$

substituting these values in the first equation we find (16).

107. Kepler in his **second law** had established the empirical fact that *the orbits of the planets are ellipses, with the sun at one of the foci.*

From this Newton concluded that the law of acceleration must be that of the inverse square of the distance from the sun. Our equation (16) enables us to draw this conclusion. The polar equation of an ellipse referred to focus and major axis is

$$r = \frac{l}{1 + e\cos\theta}, \quad i.\ e. \quad u = \frac{1}{l} + \frac{e}{l}\cos\theta,$$

where $l = b^2/a = a(1 - e^2)$; a, b being the semi-axes, l the semi-latus rectum, and e the eccentricity. Hence

$$\frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta, \quad \frac{d^2u}{d\theta^2} + u = \frac{1}{l};$$

so that we find

$$f(r) = \frac{c^2}{l}u^2 = \frac{c^2}{a(1 - e^2)} \cdot \frac{1}{r^2}. \tag{17}$$

108. The **third law of Kepler**, found by him likewise as an empirical fact, asserts that *the squares of the periodic times of different planets are as the cubes of the major axes of their orbits.*

From this fact Newton drew the conclusion that in the law of acceleration,

$$j \equiv f(r) = \frac{\mu}{r^2},$$

the constant μ has the same value for all the planets.

Our formulæ show this as follows. Let T be the *periodic time* of any planet, *i. e.* the time of describing an ellipse whose semi-axes are a, b. Then, since the sector described in the time T is the area πab of the whole ellipse, we have by Art. 99

$$\pi ab = \tfrac{1}{2}cT.$$

Substituting in (17) the value of c found from this equation we have

$$f(r) = \frac{4\pi^2a^2b^2}{lT^2} \cdot \frac{1}{r^2} = \frac{4\pi^2a^3}{T^2} \cdot \frac{1}{r^2}.$$

Hence

$$\mu = \frac{4\pi^2 a^3}{T^2}$$

is constant by Kepler's third law.

109. Planetary motion in its simplest form is that particular case of central motion in which the acceleration is inversely proportional to the square of the distance from the center O so that

$$j = f(r) = \frac{\mu}{r^2},$$

where μ is a constant, viz., the acceleration at the distance $r = 1$ from O.

The equations of motion (12) are in this case, with O as origin,

$$\frac{d^2x}{dt^2} = -\mu\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -\mu\frac{y}{r^3}. \tag{18}$$

Combining these by the principle of energy (Arts. 103, 104), we find

$$\frac{d}{dt}\tfrac{1}{2}v^2 = -\frac{\mu}{r^3}(x\dot{x} + y\dot{y}) = -\frac{\mu}{r^3}\frac{d}{dt}\tfrac{1}{2}(x^2 + y^2) = -\frac{\mu}{r^2}\frac{dr}{dt}.$$

Integrating the differential equation $d\tfrac{1}{2}v^2 = -(\mu/r^2)dr$ we find

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{\mu}{r} - \frac{\mu}{r_0}. \tag{19}$$

110. To find the equation of the path, or *orbit*, write the equations (18) in the form

$$\ddot{x} = -\frac{\mu}{r^2}\cos\theta, \quad \ddot{y} = -\frac{\mu}{r^2}\sin\theta$$

and eliminate r^2 by means of (11):

$$\ddot{x} = -\frac{\mu}{c}\cos\theta\cdot\dot{\theta}, \quad \ddot{y} = -\frac{\mu}{c}\sin\theta\cdot\dot{\theta}.$$

Each of these equations can be integrated by itself:

$$\dot{x} - v_1 = -\frac{\mu}{c}\sin\theta, \quad \dot{y} - v_2 = \frac{\mu}{c}(\cos\theta - 1), \tag{20}$$

where v_1, v_2 are the components of the velocity when $\theta = 0$.

Multiplying by y, x, subtracting, and integrating, we find by Art. 102:

$$\left(\frac{\mu}{c} - v_2\right)x + v_1 y + c = \frac{\mu}{c}(x \cos\theta + y \sin\theta) = \frac{\mu}{c}\sqrt{x^2 + y^2}. \quad (21)$$

111. The geometrical meaning of this equation is that the radius vector $r = \sqrt{x^2 + y^2}$ drawn from the fixed point O to the moving point P is proportional to the distance of P from the fixed straight line

$$\left(\frac{\mu}{c} - v_2\right) x + v_1 y + c = 0. \quad (22)$$

It represents, therefore, a conic section having O for a focus and the line (22) for the corresponding directrix.

The character of the conic depends on the absolute value of the ratio of the radius vector to the distance from the directrix; according as this ratio

$$\frac{c}{\mu}\sqrt{\left(\frac{\mu}{c} - v_2\right)^2 + v_1^2},$$

is < 1, $= 1$, or > 1, the conic will be an ellpise, a parabola, or a hyperbola. This criterion can be simplified. Multiplying by μ/c and squaring, we have

$$-\frac{2\mu v_2}{c} + v_2^2 + v_1^2 \lesseqqgtr 0,$$

or since $v_1^2 + v_2^2 = v_0^2$ and $c = r_0 v_0 \sin\psi_0 = r_0 v_2$:

$$v_0^2 \lesseqqgtr \frac{2\mu}{r_0}. \quad (23)$$

112. If polar co-ordinates be introduced in (21), the equation of the orbit assumes the form

$$\frac{1}{r} = \frac{\mu}{c^2} + \left(\frac{v_2}{c} - \frac{\mu}{c^2}\right) \cos\theta - \frac{v_1}{c} \sin\theta,$$

or putting $(cv_2 - \mu)/c^2 = C \cos\alpha$, $v_1/c = C \sin\alpha$,

$$\frac{1}{r} = \frac{\mu}{c^2} + C \cos(\theta + \alpha). \quad (24)$$

This equation might have been obtained directly by integrating (16), which in our case, with $f(r) = \mu/r^2$, reduces to

$$\frac{d^2}{d\theta^2}\frac{1}{r} + \frac{1}{r} = \frac{\mu}{c^2};$$

the general integral of this differential equation is of the form (24), C and α being the constants of integration.

Equation (24) represents a conic section referred to the focus as origin and a line making an angle α with the focal axis as polar axis.

113. Exercises.

(1) A point moves in a circle; if the acceleration be constant in direction, what is its magnitude?

(2) A point describes a circle; if the acceleration be constantly directed towards the center, what is its magnitude?

(3) A point has a central acceleration proportional to the distance from the center and directed away from the center; find the equation of the path.

(4) A point P is subject to two accelerations, $\mu^2 \cdot O_1P$ directed toward the fixed point O_1, and $\mu^2 \cdot O_2P$ directed away from the fixed point O_2. Show that its path is a parabola.

(5) A point P describes an ellipse owing to a central acceleration $f(r) = \mu/r^2$ directed toward the focus S. Its initial velocity v_0 makes an angle ψ_0 with the initial radius vector r_0. Determine the semi-axes a, b of the ellipse in magnitude and position.

(6) Find the law of acceleration when the equation of the orbit is $r^n = q^n/(1 + e \cos n\theta)$, e being positive, and investigate the particular cases $n = 1$, $n = 2$, $n = -1$, $n = -2$.

(7) Find the law of the central acceleration directed to the origin under whose action a point will describe the following curves: (a) the spiral of Archimedes $r = a\theta$; (b) the hyperbolic spiral $\theta r = a$; (c) the logarithmic or equiangular spiral $r = ae^{n\theta}$; (d) the curve $r = a \cos n\theta$.

(8) A point moves in a circle and has its acceleration directed towards a point on the circumference. Find the law of acceleration.

(9) The acceleration of a point is perpendicular to a given plane and inversely proportional to the cube of the distance from the plane. Determine its motion.

(10) A point moves in a semi-ellipse with an acceleration perpendicular to the axis joining the ends of the semi-ellipse. Determine the law of acceleration and the velocity.

CHAPTER IV.

VELOCITIES IN THE RIGID BODY.

1. Geometrical discussion.

114. The velocities of the various points of a rigid body, at any instant, are in general different, both in magnitude and direction; *i. e.* they are different *vectors*; but they are not independent of each other

In particular, it is clearly possible (*i. e.* compatible with the rigidity of the body) that the velocities of all points, at the given instant, are equal vectors. The instantaneous state of motion of the body is then called a **translation**; it is fully determined by the velocity vector of any one point of the body, and this is called the *velocity of translation*, or *linear velocity, of the body*. Comp. Art. 30.

The ideas of absolute and relative velocity and of composition and resolution of velocities apply to the velocity of translation of a rigid body just as they apply to the linear velocity of a point (comp. Arts. 38, 40, 41).

115. As the position of a rigid body is fully determined by the positions of any three of its points, O_1, O_2, O_3, not in a straight line, it is clear that if any three such points have zero velocity at any instant, all points of the body must have zero velocity at that instant. The body is then said to be *instantaneously at rest*.

It may also be regarded as geometrically obvious that if any two points O_1, O_2 of a rigid body have zero velocity, all points of the line l joining O_1 and O_2 must have zero velocity and hence (unless all points of the body have zero velocity)

the velocity of every point P of the body is normal to the plane (l, P) and proportional to the distance of P from l (comp. Art. 31). The instantaneous state of motion of the body is then called a **rotation**; the line l is called the *instantaneous axis of rotation*; and the common factor of proportionality ω of the velocities is called the *angular velocity*.

It is convenient to think of the rotation as represented geometrically by a vector of length ω, laid off on the axis of rotation l, in a sense such that the rotation appears counter-clockwise as seen from the arrowhead of the vector (Fig. 5, Art. 31). Such a vector confined to a definite straight line is called a *localized vector*, or **rotor**. The rotor ω fully characterizes the instantaneous state of motion of the body since the velocity of every point of the body can be found from it as we shall see in Art. 118.

116. *The instantaneous state of motion of a rigid body one of whose points is fixed*, if not a state of rest, *is a rotation*. For, it can be shown that if one point O of the body has zero velocity there exists a line l through O all of whose points have zero velocity. An analytical proof is given in Art. 128. Geometrically the proposition can be proved as follows.

Observe first that *in any motion of a rigid line the velocities of all points of the line must have equal projections on the line*; this follows directly from the rigidity of the line. Hence if the velocity of any point of the line is normal to the line or zero the velocities of all points of the line must be either normal to the line or zero.

Now consider a rigid body of which one point O has zero velocity, and let P_1, P_2 be any two points of the body, not in line with O. The velocities of P_1, P_2 must be normal to OP_1, OP_2, respectively. If the velocity of either of these points were zero, the line joining this point to O would be

the required axis of rotation. We assume therefore that these velocities are both different from zero. We can also assume that these velocities are not parallel; for if they happened to be so we could replace one of the two points by a point whose velocity is not parallel to those of P_1 and P_2; otherwise the motion would be a translation which is impossible for a body with a fixed point.

It follows that the planes through P_1, P_2, normal respectively to the velocities of P_1, P_2, must intersect in a line l which of course passes through O; this line l is the axis of rotation. For, any point P of l must have a velocity normal to PO, and at the same time normal to both PP_1 and PP_2; this means that the velocity of P is zero.

117. Composition of intersecting rotors. A rigid body C may have, at a given instant, an angular velocity ω_1 about an axis l_1, while the body of reference B to which l_1 belongs rotates at the same instant with angular velocity ω_2 about an axis l_2 belonging to a fixed body A. We then say that, with respect to A, the body C has the *simultaneous angular velocities* ω_1 about l_1 and ω_2 about l_2.

If the axes l_1, l_2 intersect, say at O, the instantaneous motion of C with respect to A is a rotation about an axis l passing through O such that

$$\frac{\sin l_1 l}{\omega_2} = \frac{\sin l l_2}{\omega_1} = \frac{\sin l_1 l_2}{\omega},$$

with an angular velocity

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos l_1 l_2}.$$

This proposition, known as the **parallelogram of angular velocities,** means simply that two simultaneous angular velocities ω_1, ω_2, about intersecting axes l_1, l_2, are together

equivalent to a single angular velocity ω about l, whose rotor ω is the geometric sum of the rotors ω_1, ω_2 (Fig. 27). The proof is as follows.

The linear velocity of any point P of the body has two components, $\omega_1 r_1$ and $\omega_2 r_2$, where r_1, r_2 are the perpendiculars let fall from P on the axes l_1, l_2. These components lie in the same line only for the points of the plane (l_1, l_2); and they are equal and opposite only for the points on the diagonal of the parallelogram constructed on ω_1, ω_2 as sides. All the points of this diagonal having the velocity zero, this line is the axis of rotation. The above equations follow at once from the parallelogram construction.

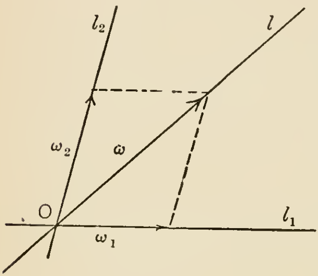

Fig. 27.

The proposition is readily extended to the composition of three or more angular velocities about axes passing through the same point. Conversely, the proposition is used to resolve a rotor ω along lines through any point of its line l. The resolution of ω along three rectangular lines through such a point into the components ω_x, ω_y, ω_z, so that $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$, is used very often.

118. *In the case of rotation*, of angular velocity ω, about an axis l, if the motion be referred to rectangular axes, with the origin O on l, we can find *the components v_x, v_y, v_z of the velocity v of any point $P(x, y, z)$ of the body*, by replacing ω by its components ω_x, ω_y, ω_z (Art. 117). It then follows from Art. 48, Ex. 1, that ω_x produces at P a velocity whose components are 0, $-\omega_x z$, $\omega_x y$; similarly, ω_y gives the components $\omega_y z$,

0, $-\omega_y x$; and ω_z gives $-\omega_z y, \omega_z x, 0$. Adding the components having the same direction, we find

$$v_x = \omega_y z - \omega_z y, \quad v_y = \omega_z x - \omega_x z, \; v_z = \omega_x y - \omega_y x.$$

119. By Art. 115, the velocity v of P (Fig. 28) is normal to the plane (l, P) and equal to $\omega \cdot CP$, where C is the foot of the perpendicular let fall from P on l. Putting $OP = r$, $\measuredangle\, COP = \phi$, so that $CP = r \sin\phi$, we find

$$v = \omega r \sin\phi.$$

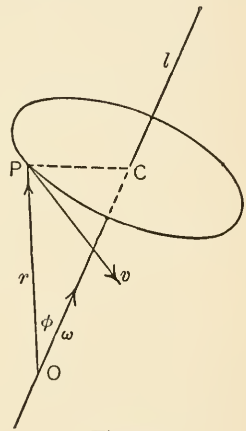

Fig. 28.

This is numerically equal to the area of the parallelogram constructed on the vectors ω and r.

In vector analysis the area of the parallelogram of any two vectors a, b is represented by a vector c, of magnitude $c = ab \sin\phi$ (ϕ being the angle between a and b), drawn at right angles to both a and b, in such a sense that a, b, c form a right-handed set. This vector c is called the *cross-product* (vector product, external product) of a and b and is denoted by $a \times b$ (read a cross b).

It then appears that *the linear velocity v of P in our case* (Art. 118) *is the cross-product of the angular velocity ω into the radius vector r of P*:

$$v = \omega \times r.$$

If the components of a, b, c with respect to rectangular axes are denoted by subscripts x, y, z, it is shown in vector analysis that

$$c_x = a_y b_z - a_z b_y, \quad c_y = a_z b_x - a_x b_z, \quad c_z = a_x b_y - a_y b_x.$$

This means that the vector equation $v = \omega \times r$ is equivalent to the last three equations of Art. 118.

120. *The most general instantaneous state of motion of a rigid body consists of a simultaneous rotation and translation.* For, whatever the state of motion, if we impose on the whole body a velocity of translation $-u$ equal and opposite to the linear velocity u of any one of its points O, so as to reduce the velocity of O to zero, we have a body in a state of rotation (Art. 116). Hence the state of motion of a rigid body can always be regarded as consisting of a velocity of translation u equal to the velocity of any one of its points O, together with an angular velocity ω about an axis l through O.

121. The **composition of parallel rotors** can be regarded as a limiting case of that of intersecting rotors (Art. 117); but it is best to prove the corresponding formulæ directly. Angular velocities about parallel axes occur, in particular, in the case of *plane* motion of a rigid body (see Art. 132).

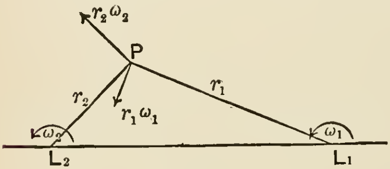

Fig. 29.

Consider a body turning with angular velocity ω_1 about an axis l_1 (passing through the point L_1, Fig. 29, at right angles to the plane of this figure) and at the same time with angular velocity ω_2 about an axis l_2 (through L_2) parallel to l_1.

Any point P of the body receives from ω_1 a linear velocity $\omega_1 r_1$ perpendicular to L_1P and from ω_2 a linear velocity $\omega_2 r_2$ perpendicular to L_2P; the resultant of these two is the total velocity of P. The two components $\omega_1 r_1$ and $\omega_2 r_2$ fall into the same straight line only for points in the plane $(l_1 l_2)$, and their resultant will be zero only for those points of this plane which divide the distance between l_1 and l_2 in the inverse

ratio of ω_1 and ω_2. In other words, the points of zero velocity lie on a straight line l, parallel to l_1 and l_2, in the plane $(l_1 l_2)$, so situated that if L be its intersection with $L_1 L_2$, we have

$$\omega_1 \cdot L_1 L = \omega_2 \cdot L L_2.$$

To find the angular velocity ω of the rotation about l consider a particular point, for instance L_2; its linear velocity being due entirely to ω_1 about l_1 is $= \omega_1 \cdot L_1 L_2$, but it can also be regarded as due to ω about l; hence

$$\omega_1 \cdot L_1 L_2 = \omega \cdot L L_2.$$

These two relations give

$$\frac{L_1 L}{\omega_2} = \frac{L L_2}{\omega_1} = \frac{L_1 L_2}{\omega},$$

and as $L_1 L + L L_2 = L_1 L_2$, we also have

$$\omega = \omega_1 + \omega_2.$$

Thus, *the resultant of two angular velocities ω_1, ω_2 about parallel axes l_1, l_2 is an angular velocity ω equal to their algebraic sum, $\omega = \omega_1 + \omega_2$, about a parallel axis l that divides the distance between l_1, l_2 in the inverse ratio of ω_1 and ω_2.* The only exceptional case, viz. when $\omega_1 + \omega_2 = 0$, is discussed in Art. 122.

Conversely, an angular velocity ω about an axis l can always be replaced by two angular velocities ω_1, ω_2 whose sum is equal to ω and whose axes l_1, l_2 are parallel to l and so selected that l divides the distance between l_1, l_2 inversely as ω_1 is to ω_2.

122. The resulting axis lies between L_1 and L_2 when the components ω_1, ω_2 have the same sense; when they are of opposite sense, it lies without, on the side of the greater one of these components.

If ω_1 and ω_2 are equal and opposite, say $\omega_1 = \omega$, $\omega_2 = -\omega$, the resulting axis would lie at infinity. Two such equal and

opposite angular velocities about parallel axes are said to form a **rotor-couple**; its effect on the rigid body is that of a velocity of translation $v = L_1L_2 \cdot \omega = p \cdot \omega$ at right angles to the plane of the axes. The distance of the rotors, $L_1L_2 = p$, is called the *arm* of the couple, and the product $p\omega = v$ its *moment*.

A velocity of translation v can therefore always be replaced by a rotor-couple of moment $p\omega = v$, whose axes have the distance p and lie in a plane at right angles to v.

Again, an angular velocity ω about an axis l can be replaced by an equal angular velocity ω about a parallel axis l' at the distance p from l, in combination with a velocity of translation $v = \omega p$ at right angles to the plane determined by l and l'.

It easily follows from these propositions that *the resultant of any number of velocities of translation $v, v', \cdots$, parallel to the same plane, and any number of angular velocities $\omega, \omega', \cdots$ about axes perpendicular to this plane is always a single angular velocity about an axis perpendicular to the plane or a single velocity of translation parallel to the plane.*

123. We are now prepared to represent in the most simple form *the most general state of motion of a rigid body.* We saw in Art. 120 that it can be represented by the linear velocity u of any point O of the body, together with an angular velocity ω about an axis l through O.

Let us resolve u into u_0 along l and u_1 at right angles to l (Fig. 30). In the plane through l, perpendicular to u_1, we can always (if $u_1 \neq 0$) find a line l_0 parallel to l at a distance p from l such that $p\omega = -u_1$; this line l_0 is called the **central axis**.

If we apply to the body equal and opposite angular velocities $\omega, -\omega$ about l_0, the body can be regarded as having

the angular velocity ω about l_0 and the linear velocity u_0 along l_0; for, the rotor couple formed by ω about l and $-\omega$ about l_0 is, by Art. 122, equivalent to a velocity of translation $p\omega$ equal and opposite to u_1 (comp. Art. 225).

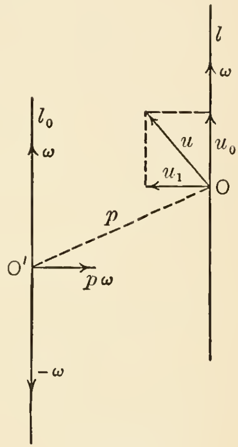

Fig. 30.

The combination of an angular velocity with a linear velocity *along the axis of rotation* is called a **twist**, or *instantaneous screw motion*. Thus, *the state of motion of a rigid body at any instant is a twist about the central axis;* it may, in particular, reduce to a mere rotation, or to a translation, or to a state of instantaneous rest.

If, as in Art. 120, we select an arbitrary point O of the body as origin of reduction, we obtain a rotor ω through O and a vector u inclined to ω at a certain angle. The rotors for different points O are always of the same magnitude, direction, and sense; but the vectors u differ in general from point to point, or rather from axis to axis. If the origin is taken on the central axis l_0, u is parallel to l_0 and has its least value, viz. u_0, the projection of u on l_0.

2. Analytical discussion.

124. In studying the motion of a rigid body *analytically* it is convenient to use two rectangular co-ordinate systems (Fig. 31), one $Oxyz$ fixed in space, the other $O_1x_1y_1z_1$ fixed in the body and moving with it. The co-ordinates x_1, y_1, z_1 of any point P of the body with respect to the moving trihedral are then constant with respect to time, while the

co-ordinates x, y, z of the same point P with respect to the fixed trihedral are functions of the time. It is assumed throughout that these functions possess first and second derivatives with respect to t.

The position of the moving trihedral at any instant is given by the co-ordinates x_0, y_0, z_0 of the origin O_1 and by

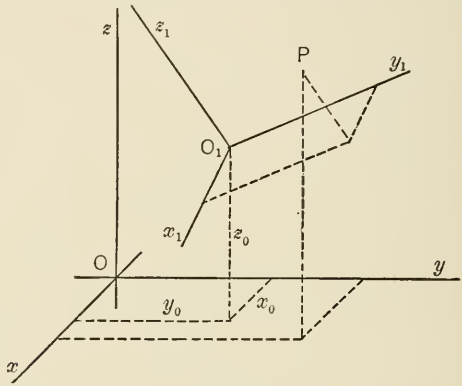

Fig. 31.

the nine direction cosines of the axes O_1x_1, O_1y_1, O_1z_1 with respect to the fixed trihedral:

	x_1	y_1	z_1
x	a_1	a_2	a_3
y	b_1	b_2	b_3
z	c_1	c_2	c_3

These 12 quantities are functions of the time.

125. The ordinary formulæ for the transformation of rectangular co-ordinates give

$$
\begin{aligned}
x &= x_0 + a_1x_1 + a_2y_1 + a_3z_1,\\
y &= y_0 + b_1x_1 + b_2y_1 + b_3z_1,\\
z &= z_0 + c_1x_1 + c_2y_1 + c_3z_1.
\end{aligned}
\tag{1}
$$

It is well known that the 9 direction cosines are connected by 6 independent relations which can be written in either one of the equivalent forms

$$
\begin{aligned}
a_1^2 + b_1^2 + c_1^2 &= 1, & a_2a_3 + b_2b_3 + c_2c_3 &= 0,\\
a_2^2 + b_2^2 + c_2^2 &= 1, & a_3a_1 + b_3b_1 + c_3c_1 &= 0,\\
a_3^2 + b_3^2 + c_3^2 &= 1, & a_1a_2 + b_1b_2 + c_1c_2 &= 0,
\end{aligned} \tag{2}
$$

or

$$
\begin{aligned}
a_1^2 + a_2^2 + a_3^2 &= 1, & b_1c_1 + b_2c_2 + b_3c_3 &= 0,\\
b_1^2 + b_2^2 + b_3^2 &= 1, & c_1a_1 + c_2a_2 + c_3a_3 &= 0,\\
c_1^2 + c_2^2 + c_3^2 &= 1, & a_1b_1 + a_2b_2 + a_3b_3 &= 0
\end{aligned} \tag{2'}
$$

The meaning of these equations readily appears from the meaning of the angles involved. Thus, the first equation expresses the fact that a_1, b_1, c_1 are the direction cosines of a line, viz. the axis O_1x_1; the last equation expresses the perpendicularity of the axes Ox, Oy; and similarly for the others.

In mechanics, the two trihedrals are generally taken as both right-handed (or both left-handed) so that they can be brought to coincidence. It is known that then the determinant of the direction cosines is $= +1$ (and not -1)

126. Differentiating the fundamental equations (1) with respect to the time t, we find for the *components of the velocity* of any point P of the rigid body *along the fixed axes:*

$$
\begin{aligned}
\dot{x} &= \dot{x}_0 + \dot{a}_1x_1 + \dot{a}_2y_1 + \dot{a}_3z_1,\\
\dot{y} &= \dot{y}_0 + \dot{b}_1x_1 + \dot{b}_2y_1 + \dot{b}_3z_1,\\
\dot{z} &= \dot{z}_0 + \dot{c}_1x_1 + \dot{c}_2y_1 + \dot{c}_3z_1.
\end{aligned} \tag{3}
$$

Notice in particular that if the motion of the body is a *translation*, the direction cosines of the moving axes are constant so that $\dot{a}_1, \cdots \dot{c}_3$ are zero; all points of the body have

then the same velocity $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$. Again if the point O_1 of the body is fixed, $\dot{x}_0, \dot{y}_0, \dot{z}_0$ are zero, and the velocity components are linear *homogeneous* functions of x_1, y_1, z_1.

127. The velocity of any point P of the body *relative to the point* O_1 has *along the fixed axes* the components:

$$\begin{aligned}\dot{x} - \dot{x}_0 &= \dot{a}_1 x_1 + \dot{a}_2 y_1 + \dot{a}_3 z_1,\\ \dot{y} - \dot{y}_0 &= \dot{b}_1 x_1 + \dot{b}_2 y_1 + \dot{b}_3 z_1,\\ \dot{z} - \dot{z}_0 &= \dot{c}_1 x_1 + \dot{c}_2 y_1 + \dot{c}_3 z_1.\end{aligned}$$

To find the components *along the moving axes* of this same relative velocity of P we have only to project the components along the fixed axes on the moving axes, which is readily done by means of the scheme of direction cosines in Art. 124. The resulting expressions

$$\begin{aligned}&(a_1\dot{a}_1 + b_1\dot{b}_1 + c_1\dot{c}_1)x_1 + (a_1\dot{a}_2 + b_1\dot{b}_2 + c_1\dot{c}_2)y_1\\ &\qquad\qquad\qquad + (a_1\dot{a}_3 + b_1\dot{b}_3 + c_1\dot{c}_3)z_1,\\ &(a_2\dot{a}_1 + b_2\dot{b}_1 + c_2\dot{c}_1)x_1 + (a_2\dot{a}_2 + b_2\dot{b}_2 + c_2\dot{c}_2)y_1\\ &\qquad\qquad\qquad + (a_2\dot{a}_3 + b_2\dot{b}_3 + c_2\dot{c}_3)z_1,\\ &(a_3\dot{a}_1 + b_3\dot{b}_1 + c_3\dot{c}_1)x_1 + (a_3\dot{a}_2 + b_3\dot{b}_2 + c_3\dot{c}_2)y_1\\ &\qquad\qquad\qquad + (a_3\dot{a}_3 + b_3\dot{b}_3 + c_3\dot{c}_3)z_1\end{aligned}$$

can be simplified very much by means of the identities (2) which give upon differentiation with respect to t:

$$\begin{aligned}\dot{a}_1 a_1 + \dot{b}_1 b_1 + \dot{c}_1 c_1 &= 0,\\ \dot{a}_2 a_2 + \dot{b}_2 b_2 + \dot{c}_2 c_2 &= 0,\\ \dot{a}_3 a_3 + \dot{b}_3 b_3 + \dot{c}_3 c_3 &= 0,\\ \dot{a}_2 a_3 + \dot{b}_2 b_3 + \dot{c}_2 c_3 &= -(a_2\dot{a}_3 + b_2\dot{b}_3 + c_2\dot{c}_3),\\ \dot{a}_3 a_1 + \dot{b}_3 b_1 + \dot{c}_3 c_1 &= -(a_3\dot{a}_1 + b_3\dot{b}_1 + c_3\dot{c}_1),\\ \dot{a}_1 a_2 + \dot{b}_1 b_2 + \dot{c}_1 c_2 &= -(a_1\dot{a}_2 + b_1\dot{b}_2 + c_1\dot{c}_2).\end{aligned}\tag{4}$$

Denoting, for the sake of brevity, the left-hand members of the last three equations by $\omega_1, \omega_2, \omega_3$ (we shall find very soon that these are precisely the components along the moving

axes of the rotor ω) we find for the *components along the moving axes of the velocity of P relative to O_1*, the simple expressions

$$\omega_2 z_1 - \omega_3 y_1, \quad \omega_3 x_1 - \omega_1 z_1, \quad \omega_1 y_1 - \omega_2 x_1,$$

which agree (considering our present notation) with the values found in Art. 118.

128. The locus of those points of the body whose velocity relative to O_1 is zero is given by

$$\omega_2 z_1 - \omega_3 y_1 = 0, \quad \omega_3 x_1 - \omega_1 z_1 = 0, \quad \omega_1 y_1 - \omega_2 x_1 = 0,$$

i. e. by

$$\frac{x_1}{\omega_1} = \frac{y_1}{\omega_2} = \frac{z_1}{\omega_3}.$$

This is a straight line l through O_1 whose direction cosines are proportional to $\omega_1, \omega_2, \omega_3$. Hence *the motion of the body relative to O_1 is a rotation about the line l.*

To see that $\omega_1, \omega_2, \omega_3$ are the angular velocities about the axes O_1x_1, O_1y_1, O_1z_1, respectively, take O_1 as origin and the line l as axis O_1z_1; then the velocity of any point in the x_1y_1-plane has the components $-\omega_3 y_1, \omega_3 x_1, 0$; *i. e.* (Art. 48, Ex. 1) ω_3 is the angular velocity about O_1z_1; similarly for ω_1, ω_2. Comp. Art. 118. By Art. 117, the three angular velocities $\omega_1, \omega_2, \omega_3$ about O_1x_1, O_1y_1, O_1z_1 are together equivalent to the single angular velocity $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$ about the line through O_1 whose direction cosines are proportional to ω_1, ω_2, ω_3.

129. If, as in Art. 120, we denote by u the velocity of the point O_1 and by u_1, u_2, u_3 its components along the moving axes, we have for the *components v_1, v_2, v_3 of the absolute velocity of any point P (x_1, y_1, z_1) of the body along the moving axes*:

$$\begin{aligned} v_1 &= u_1 + \omega_2 z_1 - \omega_3 y_1, \\ v_2 &= u_2 + \omega_3 x_1 - \omega_1 z_1, \\ v_3 &= u_3 + \omega_1 y_1 - \omega_2 x_1; \end{aligned} \qquad (5)$$

or in vector notation

$$v = u + \omega \times r.$$

On the other hand, referring the motion to the fixed axes Ox, Oy, Oz, let x_0, y_0, z_0 be (as in Arts. 125, 126) the coordinates with respect to these axes of any point O_1 of the body, and let ω_x, ω_y, ω_z be the components of ω along the fixed axes; then the *components of the absolute velocity of any point $P(x, y, z)$ of the body along the fixed axes* are

$$\begin{aligned} \dot{x} &= \dot{x}_0 + \omega_y(z - z_0) - \omega_z(y - y_0), \\ \dot{y} &= \dot{y}_0 + \omega_z(x - x_0) - \omega_x(z - z_0), \\ \dot{z} &= \dot{z}_0 + \omega_x(y - y_0) - \omega_y(x - x_0). \end{aligned}$$

If in these formulæ we put $x = 0$, $y = 0$, $z = 0$ we obtain the components u_x, u_y, u_z (along the fixed axes) of the velocity of the origin O, regarded as a point of the moving body, viz.

$$\begin{aligned} u_x &= \dot{x}_0 - \omega_y z_0 + \omega_z y_0, \\ u_y &= \dot{y}_0 - \omega_z x_0 + \omega_x z_0, \\ u_z &= \dot{z}_0 - \omega_x y_0 + \omega_y x_0. \end{aligned}$$

By introducing these components in the preceding formulæ we obtain for the *components of the velocity v of any point P (x, y, z) of the body along the fixed axes* the simple expressions

$$\begin{aligned} v_x &= \dot{x} = u_x + \omega_y z - \omega_z y, \\ v_y &= \dot{y} = u_y + \omega_z x - \omega_x z, \\ v_z &= \dot{z} = u_z + \omega_x y - \omega_y x. \end{aligned} \qquad (6)$$

Thus the velocity v is the resultant of the velocity u of O (*i. e.* of that point of the rigid body which at the instant considered happens to coincide with the fixed origin O) and of the linear velocity arising from the rotation of angular

velocity ω about the line through O parallel to the instantaneous axis.

The equations (5) and (6) are of exactly the same form; each of these sets of equations is equivalent to the vector relation

$$v = u + \omega \times r;$$

(5) arises by projecting on the moving axes, (6) by projecting on the fixed axes.

130. We have seen (Art. 123) that the instantaneous state of motion of a rigid body is in general a twist about the central axis (in the exceptional case of translation this line lies at infinity). In the course of the motion the central axis changes its position both in space (*i. e.* relatively to the fixed trihedral $Oxyz$) and in the body (relatively to the moving trihedral $O_1x_1y_1z_1$). If the motion is continuous, the successive positions of the central axis in space will be the generators of a ruled surface S fixed in space; and the successive positions of the central axis in the body, *i. e.* the various lines of the body which in the course of time become central axes, will be the generators of a ruled surface S_1, fixed in the body and moving with it.

At any given instant these surfaces S and S_1 have the central axis corresponding to this instant in common; it can be shown that they are in contact along this common generator, so that the motion consists in a rotation about, and a sliding along, this generator. Two particular cases, that of the body with a fixed point and that of plane motion, deserve special mention.

131. Body with a fixed point. As the fixed point O has zero velocity the central axis is the instantaneous axis at O, and the velocity of translation is zero. The surfaces S, S_1

are cones with O as common vertex; and the motion can be shown to consist in the rolling of S_1 over S. This motion will be studied more fully in Chapter XVIII.

3. Plane motion.

132. If the velocities of all points of a rigid body remain parallel to a fixed plane, the motion of the body is fully determined by the motion of the cross-section made by the body in this plane. This case might be regarded as the limiting case of the motion of a body with a fixed point as this point is removed to infinity. But it is more instructive to study it directly.

Taking in the plane of the motion a set of fixed axes Ox, Oy (Fig. 32) and a set of moving axes O_1x_1, O_1y_1 we have

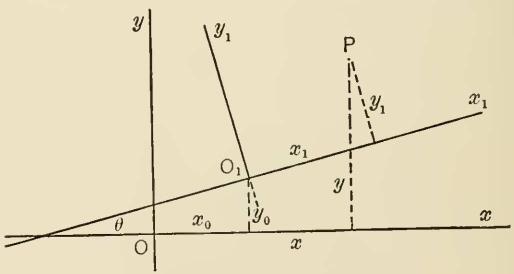

Fig. 32.

if x_0, y_0 are the co-ordinates of O_1 and θ is the angle between Ox and O_1x_1:

$$\begin{aligned} x &= x_0 + x_1 \cos\theta - y_1 \sin\theta, \\ y &= y_0 + x_1 \sin\theta + y_1 \cos\theta. \end{aligned} \tag{7}$$

133. Differentiating with respect to t we find for the components *along the fixed axes* of the velocity v of $P(x_1, y_1)$

$$\begin{aligned} \dot{x} &= \dot{x}_0 - (x_1 \sin\theta + y_1 \cos\theta)\dot{\theta} = \dot{x}_0 - \omega(y - y_0), \\ \dot{y} &= \dot{y}_0 + (x_1 \cos\theta - y_1 \sin\theta)\dot{\theta} = \dot{y}_0 + \omega(x - x_0), \end{aligned} \tag{8}$$

where $\omega = \dot{\theta}$. The velocity of O_1 has the components $\dot{x}_0$, $\dot{y}_0$; the velocity of P *relative to* O_1 has the components $-\omega(y - y_0)$, $\omega(x - x_0)$, *i. e.* it can be regarded as due to a rotation of angular velocity ω about O_1 (Art. 48, Ex. 1). The instantaneous motion of the plane section of the body consists therefore of a *translation* of velocity $u(\dot{x}_0, \dot{y}_0)$, equal to the velocity of O_1, and a *rotation* about O_1 of angular velocity $\omega = \dot{\theta}$.

Now, excluding the case of pure translation when $\omega = 0$, we can find in the plane, at any instant, a point C of zero velocity, *i. e.*, such that

$$\dot{x} = \dot{x}_0 - \omega(y - y_0) = 0, \quad \dot{y} = \dot{y}_0 + \omega(x - x_0) = 0.$$

This point C, the intersection of the central axis with the plane, is called the **instantaneous center**; its co-ordinates $\bar{x}$, $\bar{y}$ are evidently

$$\bar{x} = x_0 - \frac{\dot{y}_0}{\omega}, \quad \bar{y} = y_0 + \frac{\dot{x}_0}{\omega}. \tag{9}$$

Hence, *the instantaneous state of motion, in the case of plane motion, is either a pure translation or a pure rotation about the instantaneous center.*

It follows that (excepting the case of translation), at any instant, the velocity of every point P is normal to the radius vector CP and equal to ω times CP. Conversely, if the directions of motion of any two points P_1, P_2 are known, the instantaneous center C can in general be found as the intersection of the perpendiculars through P_1, P_2 to these directions.

134. In plane motion the ruled surfaces S, S_1 (Art. 130) are cylinders. Instead of these cylinders it suffices to consider their curves of intersection, s, s_1 with the plane. The curve s is called the **fixed, or space, centrode** (path of

the center); the curve s_1 which, as will be proved in Art. 135, rolls over s is called the **moving, or body, centrode.**

Thus *any plane motion consists in the rolling of the body centrode s_1 over the space centrode s* (except in the case of translation). It is fully determined if, in addition to any particular position of these centrodes, the angular velocity ω is given as a function of the time.

The equation of the *space centrode,* referred to the fixed axes, is found by eliminating t between the equations (9).

That of the *body centrode, i. e.* of the locus of those points of the moving figure which in the course of the motion become instantaneous centers, must be referred to the moving axes O_1x_1, O_1y_1. Substituting in (7) for x, y the values (9) and solving for x_1, y_1 we find the co-ordinates $\bar{x}_1$, $\bar{y}_1$ of the instantaneous center with respect to the moving axes:

$$\begin{aligned} \bar{x}_1 &= \frac{1}{\omega}(\dot{x}_0 \sin\theta - \dot{y}_0 \cos\theta), \\ \bar{y}_1 &= \frac{1}{\omega}(\dot{x}_0 \cos\theta + \dot{y}_0 \sin\theta); \end{aligned} \tag{10}$$

the elimination of t gives the body centrode.

135. To prove that, as stated in Art. 134, the body centrode s_1 rolls over the space centrode s it suffices to show that these curves have at the instantaneous center C not only a common point but a common tangent; in other words, that the slopes m, m_1 of s, s_1 at C are equal. These slopes can be found from the equations (9) and (10). From (9) we find by differentiating with respect to t:

$$m = \frac{\dot{\bar{y}}}{\dot{\bar{x}}} = \frac{\omega\ddot{x}_0 + \omega^2\dot{y}_0 - \dot{\omega}\dot{x}_0}{-\omega\ddot{y}_0 + \omega^2\dot{x}_0 + \dot{\omega}\dot{y}_0}.$$

Without loss of generality we may, at the instant considered,

let the moving axes coincide with the fixed axes and take the origin at the instantaneous center so that x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ are zero; we then find:

$$m = -\frac{\ddot{x}_0}{\ddot{y}_0}.$$

From (10) we find similarly

$$m_1 = \frac{\dot{y}_1}{\dot{x}_1} = \frac{\omega(\ddot{x}_0 \cos\theta + \ddot{y}_0 \sin\theta) + \omega^2(-\dot{x}_0 \sin\theta + \dot{y}_0 \cos\theta) - \dot{\omega}(\dot{x}_0 \cos\theta + \dot{y}_0 \sin\theta)}{\omega(\ddot{x}_0 \sin\theta - \ddot{y}_0 \cos\theta) + \omega^2(\dot{x}_0 \cos\theta + \dot{y}_0 \sin\theta) - \dot{\omega}(\dot{x}_0 \sin\theta - \dot{y}_0 \cos\theta)};$$

and, taking the axes as above, since $\dot{x}_0$, $\dot{y}_0$, θ are zero:

$$m_1 = -\frac{\ddot{x}_0}{\ddot{y}_0}.$$

Hence $m = m_1$, *i. e.* the curves s, s_1 have a common tangent at the instantaneous center.

It appears, moreover, that *this tangent is normal to the acceleration of the instantaneous center.* Thus, in the case of a circle rolling over a straight line, where s is the line, s_1 the circle, the acceleration of the point of contact is normal to the line.

It should be observed that the equations (9) are the parameter equations of the fixed centrode, the parameter being t; hence the t-derivatives $\dot{x}$, $\dot{y}$, used above in forming m, are not the components of the velocity of the instantaneous center *as a point of the moving figure* (these velocities are zero), but those of *the velocity*, say w, *with which* the *instantaneous center proceeds along the curve* s. Similarly the quantities $\dot{x}_1$, $\dot{y}_1$, used in forming m_1, are the components, along the moving axes, of the same velocity w.

136. This velocity w with which the instantaneous center C changes its position along the centrodes s, s_1 is connected

by a simple relation with the angular velocity ω and the radii of curvature ρ, ρ_1 of s, s_1 at C, viz.

$$\frac{\omega}{w} = \frac{1}{\rho} - \frac{1}{\rho_1}.$$

To prove this let C (Fig. 33) be the position of the instantaneous center at the time t, C' its position in the fixed plane

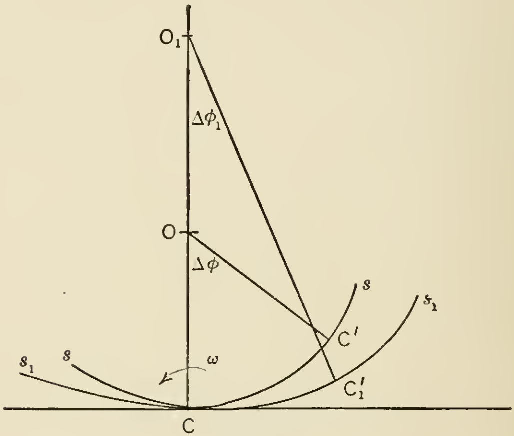

Fig. 33.

and C_1' its position in the moving figure at the time $t + \Delta t$. Then, denoting by Δs, Δs_1 the equal arcs CC', CC_1', we have as definition of w:

$$w = \lim_{\Delta t=0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t=0} \frac{\Delta s_1}{\Delta t}.$$

On the other hand, if $\Delta\theta$ is the angle through which any line of the figure turns in the time Δt we have

$$\omega = \lim_{\Delta t=0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

The motion that takes place in the interval Δt carries the

point C_1' to the position C' and brings the normal to s_1 at C_1' to coincidence with the normal to s at C'; these normals include therefore the angle $\Delta\theta$. Hence if $\Delta\varphi$, $\Delta\varphi_1$ are the angles that these normals make with the common normal at C we have $\Delta\theta = \Delta\varphi - \Delta\varphi_1$; dividing by $\Delta s = \Delta s_1$ and passing to the limit we find for the left-hand member

$$\lim_{\Delta t=0} \frac{\Delta\theta}{\Delta s} = \lim_{\Delta t=0} \frac{\Delta\theta}{\Delta t}\frac{\Delta t}{\Delta s} = \frac{\omega}{w},$$

provided $\lim \Delta s/\Delta t = w$ is $\neq 0$. In the right-hand member, the limits of $\Delta\varphi/\Delta s$ and $\Delta\varphi_1/\Delta s_1$ are clearly the curvatures of s and s_1 at C; hence

$$\frac{\omega}{w} = \frac{1}{\rho} - \frac{1}{\rho_1}.$$

It is easily seen that this formula holds even when the centers of curvature lie on opposite sides of the tangent, provided we take ρ_1 then negative. The counterclockwise sense of ω is taken as positive, and w is taken positive if the normal at C turns counterclockwise in passing to its new position through C'.

137. Exercises.

(1) *A plane figure moves in its plane so that two of its points A, B* (Fig. 34) *move along two perpendicular straight lines Ox, Oy.*

By Art. 133, the instantaneous center C is found as the intersection of the perpendiculars at A to Ox and at B to Oy. As AB is of constant length it follows readily that the space centrode is a circle of radius $AB = 2a$ about O. As $OC = AB$ it follows that the body centrode is a circle of diameter $OC = 2a$. Hence the motion can also be brought about by the rolling of a circle of radius a within a circle of twice this radius. Taking the midpoint O_1 of AB as origin and O_1A as axis O_1x_1 of the set of moving axes, and denoting by ϕ the angle BAO, we have for the co-ordinates of any point $P(x_1, y_1)$ of the moving figure:

$$x = (a + x_1)\cos\phi + y_1 \sin\phi,$$
$$y = (a - x_1)\sin\phi + y_1 \cos\phi.$$

Eliminating ϕ we find as equation of the path of P, referred to the fixed axes:

$$\left[\frac{y_1x - (a + x_1)y}{x_1^2 + y_1^2 - a^2}\right]^2 + \left[\frac{y_1y - (a - x_1)x}{x_1^2 + y_1^2 - a^2}\right]^2 = 1,$$

i. e.

$$[(a - x_1)^2 + y_1^2]x^2 - 4ay_1xy + [(a + x_1)^2 + y_1^2]y^2 = (x_1^2 + y_1^2 - a^2)^2.$$

This is an ellipse referred to its center. Show that O_1 describes a circle, and that every point on the circle about AB as diameter describes a

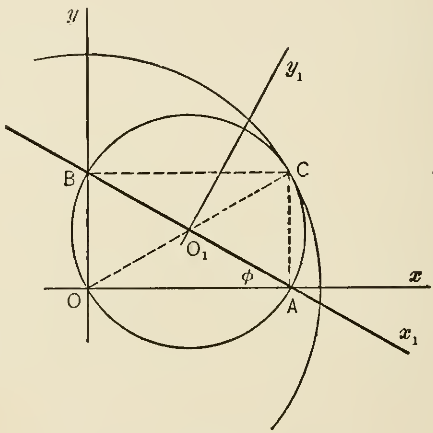

Fig. 34.

straight line through O. Show that the velocity of P is $v = [a^2 + x_1^2 + y_1^2 - 2a(x_1 \cos 2\phi + y_1 \sin 2\phi)]^{\frac{1}{2}}\dot{\phi}$; hence find the velocities of B and O_1 when A moves uniformly.

(2) *A point A of the figure moves along a fixed straight line l while a line of the figure, l_1, containing the point A, always passes through a fixed point B* (Fig. 35).

The fixed point B may be regarded as the limit of a circle which the line l_1 is to touch. The instantaneous center is therefore the intersection C of the perpendiculars erected at A to l and at B to l_1.

The fixed centrode is a parabola whose vertex is B. To prove this take the fixed line l as axis Oy, the perpendicular OB to it drawn through

the fixed point B as axis Ox. Then, putting $OBA = \phi$ and $OB = a$, we have for C:

$$x = a + y \tan\phi, \quad y = a \tan\phi,$$

whence $x - a = y^2/a$, or, with B as origin and parallel axes, $y^2 = ax$. The proportion $y/x = a/y$ also follows directly from the similar triangles BDC and AOB.

The equation of the body centrode, for A as origin, AB as polar axis, is $r \cos^2\theta = a$, or in cartesian co-ordinates $a^2(x_1^2 + y_1^2) = x_1^4$.

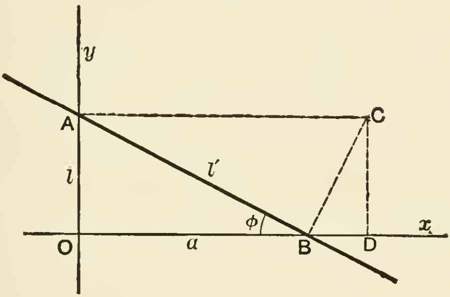

Fig. 35.

The points of l_1 are readily seen to describe conchoids; hence show how to construct the normal at any point of a conchoid.

(3) A wheel rolls on a straight track; find the direction of motion of any point on its rim. What are the centrodes?

(4) Find the equations of the centrodes when a line l_1 of a plane figure always touches a fixed circle while a point A of l_1 moves along a fixed line l.

(5) Show that, in Ex. (4), the centrodes are parabolas when the fixed circle touches the fixed line.

(6) Two straight lines l_1, l_2 of a plane figure constantly pass each through a fixed point O_1, O_2; investigate the motion.

(7) Four straight rods are jointed so as to form a plane quadrilateral $ABCD$ with invariable sides and variable angles. One side AB being fixed, investigate the motion of the opposite side; construct the centrodes graphically.

(8) A right angle moves so that one side always passes through a fixed point A, while a point B on the other side, at the distance a from

the vertex, moves along a fixed line from which the fixed point A has the distance a; determine the centrodes.

(9) If the quadrilateral of Ex. (7) be a parallelogram show that any point rigidly connected with the side opposite the fixed side describes a circle.

(10) One point A of a plane figure describes a circle while another point B moves on a straight line passing through the center O of the circle. Find the centrodes and the path of the midpoint of AB. Show how to construct the velocity of B when that of A is known.

(11) Two points P_1, P_2 of a plane figure move on two fixed circles described with radii r_1, r_2 about O_1, O_2; show that the angular velocities ω_1, ω_2 of O_1P_1, O_2P_2 about O_1, O_2 are inversely proportional to O_1M, O_2M, M being the point of intersection of O_1O_2 with P_1P_2.

(12) Given the magnitudes v_1, v_2 of the velocities of two points P_1, P_2 of a plane figure, and the angle (v_1, v_2) formed by their directions; find the instantaneous center C and the angular velocity ω about C.

CHAPTER V.

ACCELERATIONS IN THE RIGID BODY.

138. *The components, along the fixed axes, of the acceleration j of any point $P(x, y, z)$ of a rigid body* are found by differentiating with respect to t the equations (3) of Art. 126; this gives:

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 + \ddot{a}_1 x_1 + \ddot{a}_2 y_1 + \ddot{a}_3 z_1, \\ \ddot{y} &= \ddot{y}_0 + \ddot{b}_1 x_1 + \ddot{b}_2 y_1 + \ddot{b}_3 z_1, \\ \ddot{z} &= \ddot{z}_0 + \ddot{c}_1 x_1 + \ddot{c}_2 y_1 + \ddot{c}_3 z_1. \end{aligned} \tag{1}$$

But we obtain expressions that are more easily interpretable by differentiating the equations (6) of Art. 129. The first of these equations gives

$$\ddot{x} = \dot{u}_x + \omega_y \dot{z} - \omega_z \dot{y} + \dot{\omega}_y z - \dot{\omega}_z y,$$

or, replacing $\dot{y}$, $\dot{z}$ by their values from (6), Art. 129,

$$\ddot{x} = \dot{u}_x + \omega_y u_z - \omega_z u_y + \omega_y \omega_x y - \omega_y^2 x - \omega_z^2 x + \omega_z \omega_x z + \dot{\omega}_y z - \dot{\omega}_z y.$$

Adding and subtracting $\omega_x^2 x$, observing that $\omega_x^2 + \omega_y^2 + \omega_z^2 = \omega^2$, and writing down the expressions for $\ddot{y}$ and $\ddot{x}$ by cyclic permutation we find:

$$\begin{aligned} \ddot{x} &= \dot{u}_x + \omega_y u_z - \omega_z u_y + \omega_x(\omega_x x + \omega_y y + \omega_z z) \\ &\qquad - \omega^2 x + \dot{\omega}_y z - \dot{\omega}_z y, \\ \ddot{y} &= \dot{u}_y + \omega_z u_x - \omega_x u_z + \omega_y(\omega_x x + \omega_y y + \omega_z z) \\ &\qquad - \omega^2 y + \dot{\omega}_z x - \dot{\omega}_x z, \\ \ddot{z} &= \dot{u}_z + \omega_x u_y - \omega_y u_x + \omega_z(\omega_x x + \omega_y y + \omega_z z) \\ &\qquad - \omega^2 z + \dot{\omega}_x y - \dot{\omega}_y x. \end{aligned} \tag{2}$$

The meaning of the various terms will best appear by considering some particular cases.

139. In the case of **translation**, the direction cosines of the moving axes are constant, and hence (Art. 127) ω_x, ω_y, ω_z, ω are and remain zero. Hence the equations (2), as well as (1), reduce to

$$\ddot{x} = \dot{u}_x, \quad \ddot{y} = \dot{u}_y, \quad \ddot{z} = \dot{u}_z,$$

as is otherwise obvious from the definition of translation.

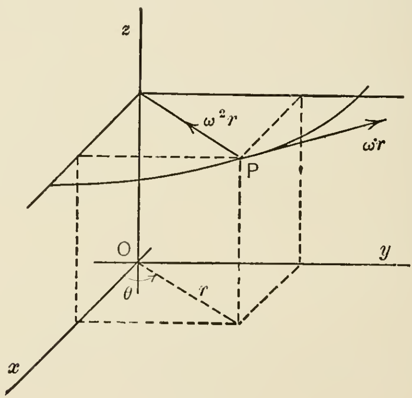

Fig. 36.

140. In the case of **rotation about a fixed axis** (Fig. 36) we can take this axis as Oz and let O_1 coincide with O, O_1z_1 with Oz. We then have

$$\begin{aligned} &\omega_x = 0, \quad \dot{\omega}_x = 0, \quad x_0 = 0, \quad u_x = 0, \quad \dot{u}_x = 0, \\ &\omega_y = 0, \quad \dot{\omega}_y = 0, \quad y_0 = 0, \quad u_y = 0, \quad \dot{u}_y = 0, \\ &\omega_z = \omega, \quad \dot{\omega}_z = \dot{\omega}, \quad z_0 = 0, \quad u_z = 0, \quad \dot{u}_z = 0; \end{aligned}$$

hence the equations (2) reduce to

$$\ddot{x} = -\omega^2 x - \dot{\omega} y, \; \ddot{y} = -\omega^2 y + \dot{\omega} x, \; \ddot{z} = \omega^2 z - \omega^2 z = 0. \quad (3)$$

The acceleration j of P is therefore parallel to the xy-plane and can be regarded as consisting of two components. Denoting by r the distance $\sqrt{x^2 + y^2}$ of P from the fixed axis, we have for one of these which is called the **normal,** or **centripetal, acceleration** j_n the components $-\omega^2 x$, $-\omega^2 y$, 0 along the fixed axes; it has therefore the magnitude

$$j_n = \omega^2 r$$

and the direction at right angles to the fixed axis, toward it. The other acceleration, called the **tangential acceleration** j_t, has the components $-\dot{\omega} y$, $\dot{\omega} x$, 0; it is tangent to the circle described by P, in the sense in which ω increases, and of magnitude

$$j_t = \dot{\omega} r.$$

These results agree of course with what is known (Art. 56, Ex. 6) about the acceleration of a point moving in a circle.

141. Let us now consider the important case of a **rigid body with a fixed point.** Taking the fixed point as fixed origin O and letting the point O_1 coincide with O we have

$$\begin{aligned} x_0 &= 0, \quad u_x = 0, \quad \dot{u}_x = 0, \\ y_0 &= 0, \quad u_y = 0, \quad \dot{u}_y = 0, \\ z_0 &= 0, \quad u_z = 0, \quad \dot{u}_z = 0, \end{aligned}$$

so that the formulæ (2) of Art. 138 reduce to

$$\begin{aligned} \ddot{x} &= \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + \dot{\omega}_y z - \dot{\omega}_z y, \\ \ddot{y} &= \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + \dot{\omega}_z x - \dot{\omega}_x z, \qquad (4) \\ \ddot{z} &= \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + \dot{\omega}_x y - \dot{\omega}_y x. \end{aligned}$$

The total acceleration j of P can here be regarded as consisting of three partial accelerations. Denoting by r the radius vector $OP = \sqrt{x^2 + y^2 + z^2}$ of P and by φ the angle between r and the rotor ω, we have

$$\omega_x x + \omega_y y + \omega_z z = \omega r \cos\varphi.$$

In vector analysis, the product $ab\cos\varphi$ of any two vectors a, b into the cosine of the angle between them is called the *dot-product* (scalar or internal product) of the vectors a and b and is written briefly $a \cdot b$ (read: a dot b). If the rectangular components of a vector are denoted by subscripts (as in Art. 119) we have

$$a \cdot b = a_x b_x + a_y b_y + a_z b_z.$$

Hence in our case

$$\omega_x x + \omega_y y + \omega_z z = \omega \cdot r.$$

Thus the first of the three partial accelerations, j_a, has along the fixed axes the components $\omega_x \omega r \cos\varphi$, $\omega_y \omega r \cos\varphi$, $\omega_z \omega r \cos\varphi$; it is therefore represented by a vector of length $j_a = \omega^2 r \cos\varphi$ whose direction is that of ω, *i. e.* of the instantaneous axis.

The second partial acceleration j_b has along the fixed axes the components $-\omega^2 x$, $-\omega^2 y$, $-\omega^2 z$; it is therefore represented by a vector of length $j_b = \omega^2 r$, along r, toward O.

The third partial acceleration j_c has along the fixed axes the components $\dot\omega_y z - \dot\omega_z y$, $\dot\omega_z x - \dot\omega_x z$, $\dot\omega_x y - \dot\omega_y x$. It is therefore, by Art. 119, the cross-product of the vectors $\dot\omega$ and r; *i. e* it has the magnitude $j_c = \dot\omega r \sin\psi$, ψ being the angle beween $\dot\omega$ and r, and it is perpendicular to both $\dot\omega$ and r, in a sense such that $\dot\omega$, r, j_c form a right-handed set.

It should be noted that each of the three partial accelerations j_a, j_b, j_c is a vector independent of the co-ordinate system, and such is of course the total acceleration j. It follows that *the components of j along the moving axes*, if those of ω are denoted by ω_1, ω_2, ω_3, will be

$$\begin{aligned}
\ddot x_1 &= \omega_1(\omega_1 x_1 + \omega_2 y_1 + \omega_3 z_1) - \omega^2 x_1 + \dot\omega_2 z_1 - \dot\omega_3 y_1,\\
\ddot y_1 &= \omega_2(\omega_1 x_1 + \omega_2 y_1 + \omega_3 z_1) - \omega^2 y_1 + \dot\omega_3 x_1 - \dot\omega_1 z_1, \qquad (4')\\
\ddot z_1 &= \omega_3(\omega_1 x_1 + \omega_2 y_1 + \omega_3 z_1) - \omega^2 z_1 + \dot\omega_1 y_1 - \dot\omega_2 x_1.
\end{aligned}$$

In vector notation we have

$$j = j_a + j_b + j_c = (\omega \cdot r_1)\omega - \omega^2 r_1 + \dot{\omega} \times r_1.$$

142. It is often convenient to combine the first two partial accelerations j_a and j_b into a single acceleration j' which is then called the **centripetal acceleration.** Now the vectors j_a, j_b both lie in the plane (l, P) determined by the instantaneous axis l (through O) and the point P (Fig. 37): $j_a = \omega^2 r \cos\varphi$ along the parallel to l through P, $j_b = \omega^2 r$ along PO; l makes with $OP = r$ the angle φ and $j_a = \omega^2 r \cos\varphi = j_b \cos\varphi$; hence the resultant j' of j_a and j_b is

$$j' = j_a \tan\varphi = j_b \sin\varphi = \omega^2 r \sin\varphi$$

along the perpendicular $PQ = r \sin\varphi = r'$ let fall from P on the instantaneous axis l. Hence finally

$$j' = \omega^2 r'.$$

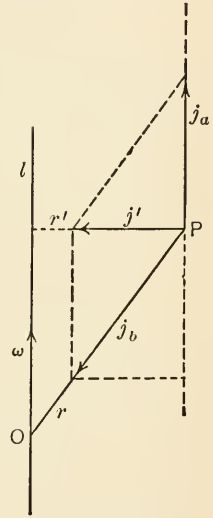

Fig. 37.

This centripetal acceleration always exists (since a body with a fixed point cannot have a motion of translation for which $\omega = 0$) except for the points on the instantaneous axis for which $r' = 0$.

143. The remaining partial acceleration j_c exists only when the rotor ω varies, in magnitude or in direction or in both.

Using the language of infinitesimals, suppose that the rotor ω in the time-element dt receives the geometrical increment $d\omega = \dot{\omega}dt$; the vector $\dot{\omega}$ may be called the *angular acceleration of the body*; its components along the fixed axes are $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$. The body has therefore the infinitesimal angular

velocities $\dot\omega_x dt$, $\dot\omega_y dt$, $\dot\omega_z dt$ about the axes Ox, Oy, Oz, respectively. These produce at $P(x, y, z)$ the infinitesimal linear velocities 0, $-\dot\omega_x z dt$, $\dot\omega_x y dt$; $\dot\omega_y z dt$, 0, $-\dot\omega_y x dt$; $-\dot\omega_z y dt$, $\dot\omega_z x dt$, 0; dividing by dt and collecting the terms we find the accelerations

$$\dot\omega_y z - \dot\omega_z y,\ \dot\omega_z x - \dot\omega_x z,\ \dot\omega_x y - \dot\omega_y x.$$

which are the components of j_c.

144. Plane motion. Taking the plane of the motion as xy-plane we have to put $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$, $\dot\omega_x = 0$, $\dot\omega_y = 0$, $\dot\omega_z = \dot\omega$, $u_z = 0$, $\dot u_z = 0$ in the equations (2) of Art. 138 so that we find

$$\ddot x = \dot u_x - \omega u_y - \omega^2 x - \dot\omega y,$$

$$\ddot y = \dot u_y + \omega u_x - \omega^2 y - \dot\omega x,$$

while $\ddot z = 0$. As $u_x = \dot x_0 + \omega y_0$, $u_y = \dot y_0 - \omega x_0$ and hence $\dot u_x = \ddot x_0 + \dot\omega y_0 + \omega \dot y_0$, $\dot u_y = \ddot y_0 - \dot\omega x_0 - \omega \dot x_0$, we find as *components of the acceleration of* $P(x, y)$ *along the fixed axes*:

$$\begin{aligned} \ddot x &= \ddot x_0 - \omega^2(x - x_0) - \dot\omega(y - y_0), \\ \ddot y &= \ddot y_0 - \omega^2(y - y_0) + \dot\omega(x - x_0). \end{aligned} \tag{5}$$

These equations are also obtained directly by differentiating the components of the velocity in plane motion, (8), Art. 133, which express that the instantaneous state of motion (unless a translation, $\omega = 0$) can be regarded as a rotation of angular velocity ω about the instantaneous center $(x_0 - \dot y_0/\omega, y_0 + \dot x_0/\omega)$.

The equations (5) show that (excepting the case of translation when $\omega = 0$, $\dot\omega = 0$) there exists at every instant a point I, the **center of acceleration,** whose acceleration is zero; its co-ordinates are

$$x_0 + \frac{\omega^2 \ddot x_0 - \dot\omega \ddot y_0}{\omega^4 + \dot\omega^2},\quad y_0 + \frac{\omega^2 \ddot y_0 + \dot\omega \ddot x_0}{\omega^4 + \dot\omega^2}. \tag{6}$$

145. If this point I of zero acceleration be taken as origin of the moving axes O_1x_1, O_1y_1 (Fig. 38), the components along

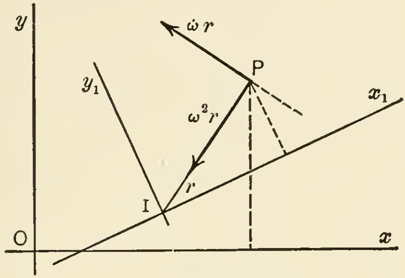

Fig. 38.

the fixed axes of the acceleration of any point $P(x, y)$ are by (5)

$$-\omega^2(x - x_0) - \dot{\omega}(y - y_0), \; -\omega^2(y - y_0) + \dot{\omega}(x - x_0).$$

The form of these expressions shows that if we put $IP = r$, *the acceleration of P can be resolved into a component $\omega^2 r$ along PI and a component $\dot{\omega} r$ at right angles to IP*; and the total acceleration of P is

$$j = r\sqrt{\omega^4 + \dot{\omega}^2}.$$

Hence, at any instant, all points on a circle about I as center have accelerations of equal magnitude and are equally inclined to their radii vectores $r = IP$; all points on a straight line through I have parallel accelerations, proportional to their radii vectores $r = IP$.

146. If any point O_1 different from I be taken as origin of the moving axes (Fig. 39) we have simply to superimpose its acceleration $j_0(\ddot{x}_0, \ddot{y}_0)$; and it appears from (5) that the acceleration of every point P can be regarded as having the three components:

$$j_0 = \text{the acceleration of } O_1,$$
$$j_1 = \omega^2 r \text{ along } PO_1,$$
$$j_2 = \dot{\omega} r \text{ at right angles to } O_1P,$$

where $r = O_1P$.

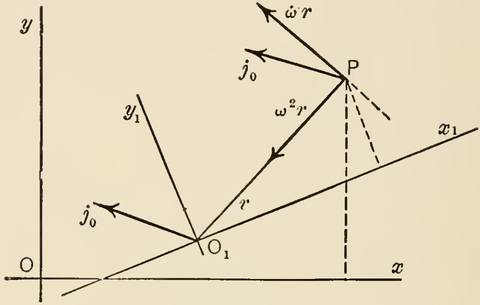

Fig. 39.

147. If, in particular, we take as origin of the moving axes the instantaneous center C and as axis O_1x_1 the common tangent of the centrodes (Fig. 40), the acceleration $\bar{j}$ of C

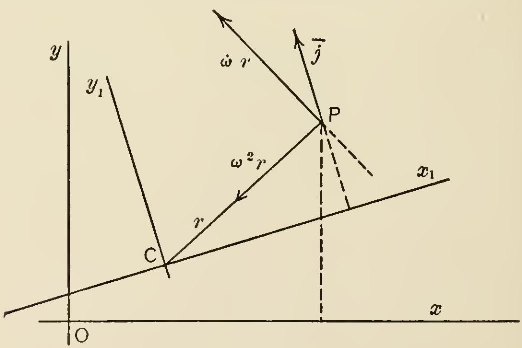

Fig. 40.

is normal to this tangent (Art. 135), and as CP is the normal to the path of P (Art. 133), the *normal* and *tangential* com-

ponents of the acceleration of P are:

$$j_n = \omega^2 r - \bar{j}\frac{y_1}{r}, \quad j_t = \dot{\omega} r + \bar{j}\frac{x_1}{r}, \tag{7}$$

where $r = CP$. Hence the loci of the points having only tangential and only normal acceleration are the circles:

$$\omega^2(x_1^2 + y_1^2) - \bar{j}y_1 = 0, \quad \dot{\omega}(x_1^2 + y_1^2) + \bar{j}x_1 = 0. \tag{8}$$

Finally, it can be shown that the *acceleration of the instantaneous center C* is

$$\bar{j} = \omega w,$$

where w is the velocity with which the instantaneous center travels along the centrodes (Art. 135). For, just as in Art. 135, we find by differentiating the equation (9) of Art. 133 and putting $\dot{x}_0 = 0$, $\dot{y}_0 = 0$ that the components of w along the fixed axes are

$$\dot{\bar{x}} = -\frac{\ddot{y}_0}{\omega}, \quad \dot{\bar{y}} = \frac{\ddot{x}_0}{\omega},$$

whence

$$\ddot{y}_0 = -\omega\dot{\bar{x}}, \quad \ddot{x}_0 = \omega\dot{\bar{y}}.$$

The acceleration of C is therefore

$$\bar{j} = \sqrt{\ddot{x}_0^2 + \ddot{y}_0^2} = \omega\sqrt{\dot{\bar{x}}^2 + \dot{\bar{y}}^2} = \omega w. \tag{9}$$

148. Exercises.

(1) A wheel of radius a rolls on a straight track; find the center of acceleration: (*a*) when the velocity v of the axis of the wheel is constant; (*b*) when the axis is uniformly accelerated, as when the wheel rolls down an inclined plane.

(2) Find the locus of the points of equal tangential acceleration.

(3) Show that the components, along the axes Cx_1, Cy_1 of Fig. 40, of the acceleration of any point are $j_1 = -\omega^2 x_1 - \dot{\omega} y_1$, $j_2 = -\omega^2 y_1 + \dot{\omega} x_1 + \bar{j}$; and hence the co-ordinates of I are $-\dot{\omega}\bar{j}/(\omega^4 + \dot{\omega}^2)$, $\omega^2\bar{j}/(\omega^4 + \dot{\omega}^2)$. Verify that these co-ordinates satisfy the equations (8); this shows that the center of acceleration is the intersection (different from C) of the circles (8).

(4) Show that the resultant of $\bar{j}$ and $\dot{\omega}r$ in Fig. 40 is an acceleration $\dot{\omega}r'$, perpendicular to $r' = HP$, where H, the *center of angular acceleration*, is the intersection of the circle of no tangential acceleration (second of the equations (8)) with the common tangent of the centrodes at C; it lies at the distance $CH = \bar{j}/\dot{\omega}$ from C. It follows that the acceleration of any point P can be resolved into two components, $\omega^2 r$ along PC and $\dot{\omega}r'$ normal to $HP = r'$.

(5) The first of the circles (8) is called the *circle of inflections;* why?

(6) Show that the diameter of the circle of inflections is the reciprocal of the difference of the curvatures of the centrodes at their point of contact.

(7) Determine the locus of the points whose acceleration at any instant is parallel: (a) to the common normal, (b) to the common tangent, of the centrodes.

CHAPTER VI.

RELATIVE MOTION.

149. In studying the motion of a point P relatively to a rigid body of reference B which is itself in motion we use, just as in Art. 124, two rectangular trihedrals, one $Oxyz$ fixed in space, the other $O_1x_1y_1z_1$ fixed in the body B and moving with it. The absolute co-ordinates x, y, z of P are connected with its relative co-ordinates x_1, y_1, z_1 by the relations (1), Art. 125; but now not only the absolute co-ordinates x, y, z but also the relative co-ordinates x_1, y_1, z_1 of P are functions of the time.

Hence, differentiating the equations (1), Art. 125, we find for the *components, along the fixed axes, of the absolute velocity v of P:*

$$\begin{aligned}\dot{x} &= \dot{x}_0 + \dot{a}_1x_1 + \dot{a}_2y_1 + \dot{a}_3z_1 + a_1\dot{x}_1 + a_2\dot{y}_1 + a_3\dot{z}_1,\\ \dot{y} &= \dot{y}_0 + \dot{b}_1x_1 + \dot{b}_2y_1 + \dot{b}_3z_1 + b_1\dot{x}_1 + b_2\dot{y}_1 + b_3\dot{z}_1,\\ \dot{z} &= \dot{z}_0 + \dot{c}_1x_1 + \dot{c}_2y_1 + \dot{c}_3z_1 + c_1\dot{x}_1 + c_2\dot{y}_1 + c_3\dot{z}_1.\end{aligned} \tag{1}$$

If the point P were rigidly connected with the body B the last three terms would be zero; hence the first four terms represent the components along the fixed axes of the so-called *body-velocity* v_b, *i. e.* the velocity of that point of the rigid body with which the point P happens to coincide at the instant considered. This also follows from the equations (3) of Art. 126.

As $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$ are the components along the moving axes of the *relative velocity* v_r of P with respect to B, the last three terms of (1) are the components along the fixed axes of this

same velocity v_r (comp. the scheme of direction cosines in Art. 124).

Thus the equations (1) are merely the analytical exprcssion of the vector equation

$$v = v_b + v_r;$$

i. e. the absolute velocity v of a point P is the geometric sum, or resultant, of the body-velocity v_b and the relative velocity v_r; comp. Art. 38.

150. Differentiating the equations (1) again with respect to t we find *the components, along the fixed axes, of the absolute acceleration j of P.*

$$\begin{aligned}
\ddot{x} &= \ddot{x}_0 + \ddot{a}_1 x_1 + \ddot{a}_2 y_1 + \ddot{a}_3 z_1 + 2(\dot{a}_1\dot{x}_1 + \dot{a}_2\dot{y}_1 + \dot{a}_3\dot{z}_1) \\
&\qquad + a_1\ddot{x}_1 + a_2\ddot{y}_1 + a_3\ddot{z}_1, \\
\ddot{y} &= \ddot{y}_0 + \ddot{b}_1 x_1 + \ddot{b}_2 y_1 + \ddot{b}_3 z_1 + 2(\dot{b}_1\dot{x}_1 + \dot{b}_2\dot{y}_1 + \dot{b}_3\dot{z}_1) \\
&\qquad + b_1\ddot{x}_1 + b_2\ddot{y}_1 + b_3\ddot{z}_1, \\
\ddot{z} &= \ddot{z}_0 + \ddot{c}_1 x_1 + \ddot{c}_2 y_1 + \ddot{c}_3 z_1 + 2(\dot{c}_1\dot{x}_1 + \dot{c}_2\dot{y}_1 + \dot{c}_{31}\dot{z}) \\
&\qquad + c_1\ddot{x}_1 + c_2\ddot{y}_1 + c_3\ddot{z}_1.
\end{aligned} \tag{2}$$

The first four terms on the right represent, by (1), Art. 138, what we may call for the sake of brevity the *body-acceleration* j_b, *i. e.* the acceleration of that point of the body of reference B with which the point P happens to coincide at the instant considered. The last three terms are the components along the fixed axes of the *relative acceleration* j_r of P whose components along the moving axes are $\ddot{x}_1$, $\ddot{y}_1$, $\ddot{z}_1$, *i. e.* of the acceleration of P relatively to the moving body B.

To interpret the middle terms, those with the factor 2, observe that by comparing Arts. 119 and 127 it appears that the velocity v of any point P of a rigid body with a fixed point O, which is a vector of length $v = \omega r \sin\varphi$, perpendicular to the rotor ω and the radius vector $r = OP$, has along the fixed axes the components

$\dot{a}_1 x_1 + \dot{a}_2 y_1 + \dot{a}_3 z_1, \quad \dot{b}_1 x_1 + \dot{b}_2 y_1 + \dot{b}_3 z_1, \quad \dot{c}_1 x_1 + \dot{c}_2 y_1 + \dot{c}_3 z_1.$
The vector that we wish to interpret has along the fixed axes the components

$$\dot{a}_1 \cdot 2\dot{x}_1 + \dot{a}_2 \cdot 2\dot{y}_1 + \dot{a}_3 \cdot 2\dot{z}_1, \ \dot{b}_1 \cdot 2\dot{x}_1 + \dot{b}_2 \cdot 2\dot{y}_1 + \dot{b}_3 \cdot 2\dot{z}_1,$$
$$\dot{c}_1 \cdot 2\dot{x}_1 + \dot{c}_2 \cdot 2\dot{y}_1 + \dot{c}_3 \cdot 2\dot{z}_1;$$

it differs from the preceding vector merely in having x_1, y_1, z_1 replaced by $2\dot{x}_1$, $2\dot{y}_1$, $2\dot{z}_1$. It represents therefore a vector of length $\omega \cdot 2v_r \sin(\omega, v_r)$, at right angles to the rotor ω and the relative velocity $v_r(\dot{x}_1, \dot{y}_1, \dot{z}_1)$, drawn in a sense such that ω, v_r, and this vector form a right-handed set. More briefly we may say (Art. 119) that this acceleration j_c, which is called variously *compound centripetal, complementary*, or *acceleration of Coriolis*, is twice the cross-product of the angular velocity ω of the body B and the relative velocity v_r of P:

$$j_c = 2\omega \times v_r.$$

Thus, *the absolute acceleration j of a point P whose motion is referred to a moving body of reference B, is the geometric sum of three accelerations, the body-acceleration j_b, the complementary acceleration j_c, and the relative acceleration j_r:*

$$j = j_b + j_c + j_r. \tag{3}$$

This proposition is known as the **theorem of Coriolis.** Applications will be given in Chap. XIX.

PART II: STATICS.

CHAPTER VII.

MASS; DENSITY.

151. Physical bodies are distinguished from geometrical configurations by the property of possessing **mass**; and the way in which this property affects their motions is studied in that part of mechanics which is called **dynamics.**

We may think of the mass, or quantity of matter, in a physical body as a certain indestructible content in the portion of space occupied by the body. By the methods of weighing explained in physics we can compare these contents of different bodies; and, taking the mass content of some particular body as the standard unit we can express the mass of every body by a single real number. We here confine ourselves to so-called gravitational masses; the number that expresses such a mass is always positive, and it remains constant in whatever way the body may move.

The student must be warned not to confound mass with weight. The weight of a body, as we shall see later, is the force with which the body is attracted by the earth; it varies, therefore, with the distance of the body from the earth's center, and would vanish completely if the earth were suddenly annihilated; while the *indestructibility of mass* is the first fundamental principle of chemistry and physics.

The modern developments in the theory of electricity may, and probably will, lead to a better understanding of

the intimate nature of mass or matter. But this would hardly affect ordinary mechanics which will always retain a wide range of applicability.

152. The **unit of mass** in the C.G.S. system (Art. 6) is the *gram*, in the F.P.S. system the *pound*. The American pound is defined (by act of Congress, 1866) as $\frac{1}{2.2046}$ of a kilogram:

$$1 \text{ lb.} = 453.597 \text{ gm.},$$

$$1 \text{ gm.} = 0.002\,204\,6 \text{ lb.}$$

The three units of *space, time*, and *mass* are called *the fundamental units of mechanics*, because with the aid of these three, the units of all other quantities occurring in mechanics can be expressed. Thus we have seen how the units of velocity and acceleration are based on those of space and time, and we shall have many more illustrations in what follows. Any unit that can be expressed mathematically by means of one or more of the fundamental units is called a *derived unit*.

153. A continuous mass of one, two, or three dimensions is said to be *homogeneous* if the masses contained in *any* two equal lengths, areas, or volumes (as the case may be) are equal. The mass is then said to be distributed *uniformly*. In all other cases the mass is said to be *heterogeneous*.

The whole mass M of a homogeneous body divided by the space V it fills is called the **density** of the mass or body; denoting density by ρ we have therefore

$$\rho = \frac{M}{V},$$

for homogeneous bodies. It follows from the definition of homogeneity that the density of a homogeneous mass can

also be found by dividing any portion ΔM of the whole mass M by the space ΔV occupied by ΔM.

In a heterogeneous body, the quotient $\Delta M/\Delta V$ is called the *average*, or *mean*, density of the portion ΔM. The limit of this average density as the space ΔV approaches zero while always containing a certain point P is called *the density of the mass M at the point P:*

$$\rho = \lim_{\Delta V=0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV}.$$

154. The *unit of density* is the density of a substance such that the unit of volume contains the unit of mass. If the units of volume and mass are selected arbitrarily, there need not of course necessarily exist any physical substance having unit density exactly. Thus in the F.P.S. system, unit density is the density of an ideal substance one pound of which would just fill a cubic foot. As a cubic foot of water has a mass of about 62½ pounds, or 1000 ounces, the density of water is about 62½ times the unit density.

The *specific density*, or *specific gravity*, of a substance, is the ratio of its density to that of water at 4° C. Let ρ be the density, ρ_1 the specific density, M the mass, V the volume of a homogeneous mass, then in British units

$$M = \rho V = 62.5 \rho_1 V.$$

In the C.G.S. system, the unit of mass has been so selected as to make the density of water equal to 1 very nearly; in other words, the unit mass (1 gram) of water, at the temperature of 4° C., fills one cubic centimeter.

In the metric system, then, there is no difference between density and specific density or specific gravity.

155. We speak in mechanics not only of three-dimensional material bodies, or *volume masses*, but also of material surfaces, or *surface masses*, and of material lines, or *linear masses*, one or two of the spatial dimensions being made to approach zero while the mass content remains finite. Thus, in a surface mass, sometimes called a shell, lamina, or membrane, a finite mass content is assigned to every finite portion

of a surface; in a linear mass, often designated as a rod, wire, or chain, a finite mass content is assigned to every finite arc of a curve.

If $d\sigma$ is the area element of the surface σ, ds the length element of the curve s, the *surface density* ρ' and the *linear density* ρ'' are defined (comp. Art. 153) by

$$\rho' = \frac{dM}{d\sigma}, \quad \rho'' = \frac{dM}{ds}.$$

156. Finally, letting all three dimensions of a physical body approach zero, while the mass content may remain finite, we arrive at the idea of the *mass-point*, or **particle**, viz. a geometrical point to which a definite mass is assigned.

As a finite physical mass is always thought of as occupying a finite space, the particle, or geometrical point endowed with a finite mass, is a pure abstraction. The importance of this conception lies not so much in its relation to the idea that physical matter is ultimately an aggregation of such points or centers possessing mass (molecules, atoms), but in the fact that for certain purposes (viz. as far as translation only is concerned) the motion of a physical solid is fully determined by the motion of a certain point in it, called the *center of mass* or **centroid**, the whole mass of the body being regarded as concentrated at this point.

CHAPTER VIII.

MOMENTS AND CENTERS OF MASS.

157. The product of a mass m, concentrated at a point P, into the distance of the point P from any given point, line, or plane is called the **moment** of this mass with respect to the point, line, or plane.

Thus, denoting by r, q, p the distance of the point P from the point O, the line l, and the plane π, respectively, we have for the moments of m with respect to O, l, π the expressions mr, mq, mp.

158. Let a system of n points, or particles, $P_1, P_2, \cdots P_n$ be given; let $m_1, m_2, \cdots m_n$ be their masses, and $p_1, p_2, \cdots p_n$ their distances from a given plane π. Then we call **moment of the system** with respect to the plane π the algebraic sum

$$m_1p_1 + m_2p_2 + \cdots + m_np_n = \Sigma mp,$$

the distances $p_1, p_2, \cdots p_n$ being taken with the same sign or opposite signs according as they lie on the same side or on opposite sides of the plane π.

It is always possible to determine one and only one distance $\bar{p}$ such that $\Sigma mp = M\bar{p}$, where $M = \Sigma m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system. If this distance $\bar{p}$ should happen to be equal to zero, the moment of the system would evidently vanish with respect to the plane π.

159. Let us now refer the points P to a rectangular set of axes and let x, y, z be their co-ordinates. Then we have for the moments of the system with respect to the co-ordinate planes yz, zx, xy, respectively:

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n = \Sigma mx = M\bar{x},$$

$$m_1y_1 + m_2y_2 + \cdots + m_ny_n = \Sigma my = M\bar{y},$$

$$m_1z_1 + m_2z_2 + \cdots + m_nz_n = \Sigma mz = M\bar{z}.$$

The point G whose co-ordinates are

$$\bar{x} = \frac{\Sigma mx}{M}, \quad \bar{y} = \frac{\Sigma my}{M}, \quad \bar{z} = \frac{\Sigma mz}{M} \tag{1}$$

is called the *center of mass*, or the **centroid**, of the system.

The centroid is, therefore, defined as *a point such that if the whole mass M of the system be concentrated at this point, its moment with respect to any one of the co-ordinate planes is equal to the moment of the system.*

160. It is easy to see that this holds not only for the co-ordinate planes but for any plane whatever. Let

$$\alpha x + \beta y + \gamma z - p_0 = 0$$

be the equation of any plane in the normal form; $\bar{p}$, p_1, p_2, $\cdots p_n$, the distances of the points G, P_1, P_2, $\cdots P_n$ from this plane. Then we wish to prove that $\Sigma mp = M\bar{p}$.

Now

$$\bar{p} = \alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} - p_0, \quad p_1 = \alpha x_1 + \beta y_1 + \gamma z_1 - p_0, \cdots;$$

hence

$$\begin{aligned}\Sigma mp &= \alpha\Sigma mx + \beta\Sigma my + \gamma\Sigma mz - p_0\Sigma m \\ &= M(\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} - p_0) = M\bar{p}.\end{aligned}$$

The centroid can therefore be defined as *a point such that its moment with respect to any plane is equal to that of the whole system, with respect to the same plane.*

It follows that *the moment of the system vanishes for any plane passing through the centroid.*

161. In the case of a continuous mass, whether it be of one, two, or three dimensions, the same reasoning will apply

if we imagine the mass divided up into elements dM of one, two, or three infinitesimal dimensions, respectively. The summations indicated above by Σ will then become integrations, so that the centroid of a continuous mass has the co-ordinates

$$\bar{x} = \frac{\int x dM}{\int dM}, \quad \bar{y} = \frac{\int y dM}{\int dM}, \quad \bar{z} = \frac{\int z dM}{\int dM}. \tag{2}$$

According as the mass is of one, two, or three dimensions, a single, double, or triple integration over the whole mass will in general be required for the determination of the moments $\int x dM$, $\int y dM$, $\int z dM$ of the mass with respect to the co-ordinate planes, as well as of the total mass $\int dM = M$.

Thus, for a mass distributed along a line or a curve we have, if ds be the line-element,

$$dM = \rho'' ds,$$

where ρ'' is the *linear density* (Art. 155); for a mass distributed over a surface-area we have, with $d\sigma$ as a surface-element,

$$dM = \rho' d\sigma,$$

where ρ' is the *surface* (or *areal*) *density;* finally, for a mass distributed throughout a volume whose element is $d\tau$,

$$dM = \rho d\tau,$$

where ρ is the *volume density.*

If the mass be distributed along a straight line, the centroid lies of course on this line, and one co-ordinate is sufficient to determine the position of the centroid. In the case of a plane area, the centroid lies in the plane and two co-ordinates determine its position; we then speak of moments with respect to lines, instead of planes.

162. If the mass be homogeneous (Art. 153), *i. e.* if the density ρ be constant, it will be noticed that ρ cancels from the numerator and denominator in the equations (2), and does not enter into the problem. Instead of speaking of a center of mass, we may then speak of a center of arc, of area, of volume. The term *centroid* is, however, to be preferred to *center*, the latter term having a recognized geometrical meaning different from that of the former.

The geometrical center of a curve or surface is a point such that any chord through it is bisected by the point; there are but few curves and surfaces possessing a center.

The centroid (Art. 160) is a point such that, for any plane passing through it, the moment of the system is equal to zero. Such a point exists for every mass, volume, area, or arc. The centroid coincides, of course, with the center, when such a center exists and the distribution of mass is uniform.

163. As soon as ρ is given either as a constant or as a function of the co-ordinates, the problem of determining the centroid of a continuous mass is merely a problem in integration. To simplify the integrations, it is of importance to select the element in a convenient way conformably to the nature of the particular problem.

Considerations of symmetry and other geometrical properties will frequently make it possible to determine the centroid without resorting to integration. Thus, in a homogeneous mass, any plane of symmetry, or any axis of symmetry, must contain the centroid, since for such a plane or line the sum of the moments is evidently zero.

It is to be observed that the whole discussion is entirely independent of the physical nature of the masses m which appear here simply as numerical coefficients, or "weights," attached to the points. Some of the masses might even be negative provided the total mass is not zero.

It will be shown later that the *center of gravity*, as well as the *center of inertia*, of a body coincides with its centroid.

164. In determining the centroid of a given system it will often be found convenient to break the system up into a number of partial systems whose centroids are either known or can be found more readily. *The moment of the whole system is* obviously *equal to the sum of the moments of the partial systems.*

Thus let the given mass M be divided into k partial masses $M_1, \cdots M_k$ so that $M = M_1 + \cdots + M_k$; let $G, G_1, \cdots G_k$ be the centroids of $M, M_1, \cdots M_k$ and $\bar{p}, \bar{p}_1, \cdots \bar{p}_k$ their distances from some fixed plane. Then we have

$$M\bar{p} = M_1\bar{p}_1 + \cdots + M_k\bar{p}_k.$$

165. The particular case of *two* partial systems occurs most frequently. We then have with reference to any plane

$$M\bar{p} = M_1\bar{p}_1 + M_2\bar{p}_2, \quad M = M_1 + M_2.$$

Letting the plane coincide successively with each of the three co-ordinate planes it will be seen that G must lie on the line joining G_1, G_2. Now taking the plane at right angles to G_1G_2 through G_1 we have

$$M \cdot G_1G = M_2 \cdot G_1G_2;$$

similarly for a plane through G_2

$$M \cdot GG_2 = M_1 \cdot G_1G_2;.$$

whence

$$\frac{G_1G}{M_2} = \frac{GG_2}{M_1} = \frac{G_1G_2}{M};$$

i. e. the centroid of the whole system divides the distance of the centroids of the two partial systems in the inverse ratio of their masses.

166. Exercises.

(1) Show that the centroid of two particles m_1, m_2 divides their distance in the inverse ratio of the masses by taking moments about the centroid.

Find the centroid:

(2) Of three masses 5, 7, 23 on a line, the mass 7 lying midway between 5 and 23.

(3) Of earth and moon, the moon's mass being 1/80 of that of the earth and the distance of their centers 240,000 miles.

(4) Of three equal particles.

(5) Of a circular arc, radius r, angle at center 2α; in particular, of a semicircle.

(6) Of the arc of a parabola, $y^2 = 4ax$, from vertex to end of latus rectum.

(7) Of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

(8) Of half the cardioid $r = a(1 + \cos\theta)$.

(9) Of an arc of the common helix $x = r\cos\theta$, $y = r\sin\theta$, $z = kr\theta$, from $\theta = 0$ to $\theta = \theta$.

(10) Of a circular arc AB, of angle $AOB = \alpha$, whose density varies as the length of the arc measured from A

(11) Show that the centroid of a triangular area is the intersection of the medians.

(12) From a square $ABCD$ of side a one corner EAF is cut off so that $AE = \frac{3}{4}a$, $AF = \frac{1}{4}a$; find the centroid of the remaining area.

(13) An isosceles right-angled triangle of sides a being cut out of the area of its circumscribed circle, find the centroid of the remaining area.

(14) Find the centroid of the surface area of a sphere between two parallel planes, by observing that this area is equal to the surface area of the circumscribed cylinder perpendicular to these planes.

(15) Show that for an area σ, bounded by a curve $y = f(x)$, the axis Ox and two ordinates, we have

$$\sigma \cdot \bar{x} = \int_{x_1}^{x_2} xy\,dx, \quad \sigma \cdot \bar{y} = \frac{1}{2}\int_{x_1}^{x_2} y^2\,dx;$$

and hence find the centroid: (*a*) of the area bounded by the parabola $y^2 = 4ax$, the axis Ox and an ordinate; (*b*) of the area between the curve $y = \sin x$ from $x = 0$ to $x = \pi$ and the axis Ox; (*c*) of a quadrant of an ellipse; (*d*) of the segment cut off from an ellipse by the chord joining the extremities of the axes.

(16) Show that for the area σ, bounded by a curve $r = f(\theta)$ and two of its radii vectores, we have

$$\sigma \cdot \bar{x} = \frac{1}{3}\int_{\theta_1}^{\theta_2} r^3 \cos\theta\,d\theta, \quad \sigma \cdot \bar{y} = \frac{1}{3}\int_{\theta_1}^{\theta_2} r^3 \sin\theta\,d\theta.$$

(17) Find the centroid of the sector of a circle, radius r, angle at center 2α.

(18) A bowl in the form of a hemisphere is closed by a circular lid of a material whose density is three times that of the bowl. Find the centroid.

(19) The cissoid $(2a - x)y^2 = x^3$ can be represented by the equations $x = 2a\sin^2\theta$, $y = 2a\sin^3\theta/\cos\theta$, where θ is the polar angle, $2a$ the distance from cusp to asymptote. Show that the centroid of the area between the curve and its asymptote divides the distance between cusp and asymptote in the ratio 5 : 1.

(20) The centroid of a rectilinear segment of length l whose linear density is proportional to the nth power of the distance from one end is at the distance $(n + 1)l/(n + 2)$ from that end. Hence show that (*a*) the centroid of a triangular area lies on the median at ⅔ the distance from the vertex to the base; (*b*) the centroid of the surface area of a cone or pyramid lies on the line joining the vertex to the centroid of the base, at ⅔ the distance from the vertex to the base; (*c*) the centroid of the volume of a cone or pyramid lies on the same line, at ¾ the distance from the vertex to the base.

(21) For a solid of revolution, generated by the revolution of the curve $y = f(x)$ about the axis of x and bounded by planes perpendicular to the axis Ox, show that the centroids of the curved surface area σ and of the volume τ are given by:

$$\sigma \cdot \bar{x}_\sigma = 2\pi \int_{x_1}^{x_2} xy\sqrt{1 + y'^2}\,dx, \quad \tau \cdot \bar{x}_\tau = \pi \int_{x_1}^{x_2} xy^2 dx.$$

(22) Find the centroid of the segment of a sphere between two parallel planes; and hence (*a*) that of a segment of height h, cut off by a plane; (*b*) that of a hemisphere; (*c*) that of a spherical sector of vertical angle 2α.

(23) Find the centroid of the paraboloid of revolution of height h, generated by the revolution of the parabola $y^2 = 4ax$ about its axis.

(24) The area bounded by the parabola $y^2 = 4ax$, the axis of x, and the ordinate $y = y_1$ revolves about the tangent at the vertex. Find the centroid of the solid of revolution so generated.

(25) The same area as in Ex. (6) revolves about the ordinate y_1. Find the centroid.

(26) Find the centroid of an octant of an ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

(27) The equations of the common cycloid referred to a cusp as origin and the base as axis of x are $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. Find the centroid: (*a*) of the arc of the semi-cycloid (*i. e.* from cusp

to vertex); (*b*) of the plane area included between the semi-cycloid and the base; (*c*) of the surface generated by the revolution of the semi-cycloid about the base; (*d*) of the volume generated in the same case.

(28) Find the centroid of a solid hemisphere whose density varies as the nth power of the distance from the center.

CHAPTER IX.

MOMENTUM; FORCE; ENERGY.

167. Let us consider a point moving with constant acceleration from rest in a straight line. We know from Kinematics (Art. 16) that its motion is determined by the equations

$$v = jt, \quad s = \tfrac{1}{2}jt^2, \quad \tfrac{1}{2}v^2 = js, \tag{1}$$

where s is the distance passed over in the time t, v the velocity, and j the acceleration, at the time t.

If, now, for the single point we substitute an m-tuple point, *i. e.* if we endow our point with the mass m, and thus make it a *particle* (see Art. 156), the equations (1) must be multiplied by m, and we obtain

$$mv = mjt, \quad ms = \tfrac{1}{2}mjt^2, \quad \tfrac{1}{2}mv^2 = mjs. \tag{2}$$

The quantities mv, mj, $\frac{1}{2}mv^2$ occurring in these equations have received special names because they correspond to certain physical conceptions of great importance.

168. *The product mv of the mass m of a particle into its velocity v is called the* **momentum,** or the *quantity of motion,* of the particle.

In observing the behavior of a physical body in motion, we notice that the effect it produces—for instance, when impinging on another body, or more generally, whenever its velocity is changed—depends not only on its velocity, but also on its mass. Familiar examples are the following: a loaded railroad car is not so easily stopped as an empty one; the destructive effect of a cannon-ball depends both on its velocity and on its mass; the larger a fly-wheel, the more difficult is it to give it a certain velocity; etc.

It is from experiences of this kind that the physical idea of mass is derived.

The fact that any change of motion in a physical body is affected by its mass is sometimes ascribed to the so-called "*inertia*," or "force of inertia," of matter, which means, however, nothing else but the property of possessing mass.

169. Momentum, being by definition (Art. 168) the product of mass and velocity, has for its *dimensions* (Art. 6),

$$MV = MLT^{-1}.$$

The *unit of momentum* is the momentum of the unit of mass having the unit of velocity. Thus in the C.G.S. system the unit of momentum is the momentum of a particle of 1 gram moving with a velocity of 1 cm. per second. In the F.P.S. system, the unit is the momentum of a particle of 1 pound mass moving with a velocity of 1 ft. per second.

To find the relations between these two units, let there be x C.G.S. units in the F.P.S. unit; then

$$x \cdot \frac{\text{gm. cm.}}{\text{sec.}} = 1 \cdot \frac{\text{lb. ft.}}{\text{sec.}};$$

hence

$$x = \frac{\text{lb.}}{\text{gm.}} \cdot \frac{\text{ft.}}{\text{cm.}},$$

or, by Art. 152 and Art. 6,

$$x = 13{,}825.7;$$

i. e. 1 F.P.S. unit of momentum = 13,825.7 C.G.S. units, and 1 C.G.S. unit = 0.000 072 33 F.P.S. units.

170. Exercises.

(1) What is the momentum of a cannon-ball weighing 200 lbs. when moving with a velocity of 1500 ft. per second?

(2) With what velocity must a railroad-truck weighing 3 tons move to have the same momentum as the cannon-ball in Ex. (1)?

(3) Determine the momentum of a one-ton ram after falling through 4 feet.

171. *The product mj of the mass m of a particle into its acceleration j is called* **force**. Denoting it by F, we may write our equations (2) in the form

$$mv = Ft, \quad s = \tfrac{1}{2}\frac{F}{m}t^2, \quad \tfrac{1}{2}mv^2 = Fs. \tag{3}$$

As long as the velocity of a particle of constant mass remains constant, its momentum remains unchanged. If the velocity changes uniformly from the value v at the time t to v' at the time t', the corresponding change of momentum is

$$mv' - mv = mjt' - mjt = F(t' - t); \tag{4}$$

hence

$$F = \frac{mv' - mv}{t' - t}. \tag{5}$$

Here the acceleration, and hence the force, was assumed constant. If F be variable, we have in the limit as $t' - t$ approaches zero:

$$F = \frac{d(mv)}{dt} = m\frac{dv}{dt}. \tag{6}$$

Instead of defining force as the product of mass and acceleration, we may therefore define it as the *rate of change of momentum with the time.*

172. Integrating equation (6), we find

$$\int_t^{t'} Fdt = mv' - mv. \tag{7}$$

The product $F(t' - t)$ of a constant force into the time $t' - t$ during which it acts, and in the case of a variable force, *the time-integral $\int_t^{t'} Fdt$,* is called the **impulse** of the force during this time.

It appears from the equations (4) and (7) that *the impulse of a force during a given time is equal to the change of momentum during that time.*

173. The idea of force is no doubt primarily derived from the sensation produced in a person by the exertion of his "muscular force." Like the sensations of light, sound, heat, etc., the sensation of exerting force is capable, in a rough way, of measurement. But the physiological and psychological phenomena attending the exertion of muscular force when analyzed more carefully are very complicated.

In popular language the term "force" is applied in a great variety of meanings. For scientific purposes it is of course necessary to attach a single definite meaning to it.

It is customary in physics to speak of force as *producing* or *generating velocity*, and to define force as the *cause of acceleration*. Thus observation shows that the velocity of a falling body increases during the fall; the cause of the observed change in the velocity, *i. e.* of the acceleration, is called the force of attraction, and is supposed to be exerted by the earth. Again, a body falling in the air, or in some other medium, is observed to increase its velocity less rapidly than a body falling *in vacuo;* a force of resistance is therefore ascribed to the medium as the cause of this change. In a similar way we speak of the expansive force of steam, of electric and magnetic forces, etc., because it is convenient to think of such agencies as producing changes of velocity.

Now, any change in the velocity v of a body of given mass m implies a change in its momentum mv; and it is this change of momentum, or rather the rate at which the momentum changes with the time, which is of prime importance in all the applications of mechanics. It is therefore convenient to have a special name for this rate of change of momentum, and that is what is called *force* in mechanics.

Thus, in using this term "force," it is not intended to assert anything as to the objective reality or actual nature of force and matter in the popular acceptation of these terms. With the ultimate causes science has nothing to do; it can observe only the phenomena themselves.

174. The definition of force (Art. 171) as the product of mass and acceleration gives the *dimensions* of force as

$$F = MJ = MLT^{-2}.$$

The *unit of force* is therefore the force of a particle of unit mass moving with unit acceleration.

Hence, in the C.G.S. system, it is the force of a particle of 1 gram moving with an acceleration of 1 cm./sec.^{2}. This unit force is called a **dyne.**

The definition is sometimes expressed in a slightly different form. We may say the dyne is the force which, acting on a gram uniformly for one second, would generate in it a velocity of 1 cm./sec.; or would give it the C.G.S. unit of acceleration; or it is the force which, acting

on *any* mass uniformly for one second, would produce in it the C.G.S. unit of momentum.

That these various statements mean the same thing follows from the fundamental formulæ $F = mj$, $v = jt$, if F, m, t, v, j be expressed in C.G.S. units.

In the F.P.S. system, the unit of force is the force of a mass of 1 lb. moving with an acceleration of 1 ft./sec.². It is called the **poundal.**

175. To find the relation between these two units, let x be the number of dynes in the poundal; then we have

$$x \cdot \frac{\text{gm. cm.}}{\text{sec.}^2} = 1 \cdot \frac{\text{lb. ft.}}{\text{sec.}^2};$$

hence, just as in Art. 169, $x = 13{,}825.7$; *i. e.* 1 poundal = 13,825.7 dynes, and 1 dyne = 0.000 072 33 poundals.

176. Another system of measuring force, the so-called **gravitation** (or engineering) **system,** is in very common use, and must be explained here.

Among the forces of nature the most common is the *force of gravity*, or the *weight*, *i. e.* the force with which any physical body is attracted by the earth. As we have convenient and accurate appliances for comparing the weights of different bodies at the same place, the idea suggests itself of selecting as unit force the weight of a certain standard mass.

In the metric gravitation system the *weight of a kilogram* has been selected as unit force; in the British gravitation system the *weight of a pound* is the unit force.

177. The system in which the units of time, length, and mass are taken as fundamental, while the unit of force (= mass times acceleration) is regarded as a *derived* unit (Art. 175), is called the *absolute* or *scientific* system, to distinguish it from the gravitation system (Art. 176) in which the units of time, length, and force are taken as fundamental, while the unit of mass (= force divided by acceleration) is a derived unit.

As the weight of a body varies from place to place with the variation of the acceleration of gravity g, the unit of force as defined in Art. 176 would not be constant. This difficulty can be avoided by defining the unit of force as the weight of a kilogram or pound *at some definite place*, say at London, or in latitude 45° at sea level. With this modification,

the gravitation system deserves the name of an *absolute* system as much as does the system in which mass is the third fundamental unit.

The general equations of mechanics are of course independent of the system of measurement adopted; they hold as well in the gravitation as in the scientific or absolute system. In the present work the language of the latter system is generally used in the text (not always in the exercises). This system, since its introduction by Gauss and Weber, has found general acceptance in scientific work.

In statics where we are mainly concerned with the *ratios* of forces and not with their absolute values it rarely makes any difference which system is used provided all forces are expressed in the same unit. And as elementary statics deals largely with the effects of gravity, the gravitation system is often used in statical problems.

178. The numerical relation between the scientific and gravitation measures of force is expressed by the equations

$$1 \text{ kilogram (force)} = 1000\, g \text{ dynes},$$
$$1 \text{ pound (force)} = g \text{ poundals},$$

where g is about 981 in metric units, and about 32.2 in British units. In most cases the more convenient values 980 and 32 may be used.

179. Exercises.

(1) What is the exact meaning of "a force of 10 tons"? Express this force in poundals and in dynes.

(2) Reduce 2,000,000 dynes to British gravitation measure.

(3) Express a pressure of 2 lbs. per square inch in kilograms per square centimeter.

(4) Show that a poundal is very nearly half an ounce, and a dyne a little over a milligram, in gravitation measure.

180. *The quantity $\frac{1}{2}mv^2$, i. e. half the product of the mass of a particle into the square of its velocity,* is called the **kinetic energy** of the particle.

Let us consider again a particle of constant mass m moving with a constant acceleration, and hence with a constant force; let v be the velocity, s the space described, at the time t; v', s' the corresponding values at the time t'. Then the last

of the three fundamental equations (see Arts. 167 and 171) gives

$$\tfrac{1}{2}mv'^2 - \tfrac{1}{2}mv^2 = F(s' - s); \tag{8}$$

hence

$$F = \frac{\tfrac{1}{2}mv'^2 - \tfrac{1}{2}mv^2}{s' - s}. \tag{9}$$

If F be variable, we have in the limit

$$F = \frac{d(\tfrac{1}{2}mv^2)}{ds} = mv\frac{dv}{ds}. \tag{10}$$

Force can therefore be defined as *the rate at which the kinetic energy changes with the space.* (Compare the end of Art. 171.)

181. Integrating the last equation (10), we find

$$\int_s^{s'} Fds = \tfrac{1}{2}mv'^2 - \tfrac{1}{2}mv^2. \tag{11}$$

The product $F(s' - s)$ of a constant force F into the space $s' - s$ described in the direction of the force, and in the case of a variable force, *the space-integral $\int_s^{s'} Fds$,* is called the **work** of the force for this space.

The equations (8) and (11) show that *the work of a force is equal to the corresponding change of the kinetic energy.*

We have here assumed that the force acts in the direction of motion of the particle. A more general definition of work including the above as a special case will be given later (Art. 261).

The ideas of energy and work have attained the highest importance in mechanics and mathematical physics within comparatively recent times. Their full discussion belongs to Kinetics (see Part III).

182. According to their definitions, both *momentum* (Art. 168) and *force* (Art. 171) may be regarded mathematically

as mere numerical multiples of velocity and acceleration, respectively. They are therefore so-called vector-quantities; *i. e.* a momentum as well as a force can be represented geometrically by a segment of a straight line of definite length, direction, and sense. Moreover, as they are referred to a particular point, viz., to the point whose mass is m, the line representing a momentum or a force must be drawn through this point; the line has therefore not only direction, but also position; *i. e. a momentum as well as a force is represented geometrically by a* **rotor** (compare Art. 115).

It follows that concurrent forces, for instance, can be compounded by geometrical addition, as will be explained more fully in Chapter X.

On the other hand, kinetic energy and work are not vector-quantities.

183. The ideas of momentum, force, energy, work, with the fundamental equations connecting them, as given in the preceding articles, form, the groundwork of the whole science of theoretical dynamics. The application of this science to the interpretation of natural phenomena gives results in close agreement with observation and experiment. It is therefore important to inquire what are the physical assumptions and experimental data on which this application of dynamics is based.

These assumptions were formulated with remarkable clearness by Sir Isaac Newton in his *Philosophiæ naturalis principia mathematica*, first published in 1687, and have since been known as **Newton's laws of motion.** As these three *axiomata sive leges motus*, as Newton terms them, are very often referred to and, at least by English writers on dynamics, are usually laid down as the foundation of the science, they are given here in a literal translation:

I. Every body persists in its state of rest or of uniform motion along a straight line, except in so far as it is compelled by impressed (*i. e.* external) forces to change that state.

II. Change of motion is proportional to the impressed moving force and takes place along the straight line in which that force acts.

III. To every action there is an equal and contrary reaction; or, the mutual actions of two bodies on one another are always equal and directed in contrary senses.

184. Some explanation is necessary to understand correctly the meaning of these laws. Indeed, Newton's laws should not be studied by themselves; they become intelligible only if taken in connection with the definitions preceding them in the *Principia*, and with the explanations and corollaries that Newton himself has appended to them.

The word "body" must be taken to mean particle; the word "motion" in the second law means what is now called momentum.

All three laws imply the idea of *force as the cause of any change of momentum* in a particle.

185. With this definition of force the first law, at least in the ordinary form of statement, for a single particle, merely states that where there is no cause there is no effect. While this law may appear superfluous to us, it was not so in the time of Newton. Kepler and Galileo, less than a century before Newton, were the first to insist more or less clearly on this so-called **law of inertia**, viz. that there is no intrinsic power or tendency in moving matter to come to rest or to change its motion in any way.

186. The second law gives as the measure of a constant force the amount of momentum generated in a given time (see Art. 171); it can be called the **law of force**. If force be defined as the cause of any change of momentum, the second law follows naturally by assuming, as is usually done, that the effect is proportional to the cause.

The first two laws may thus be regarded from the mathematical point of view as nothing but a definition of force; but they are certainly meant to emphasize the physical fact that the assumed definition of force is not arbitrary, but based on the characteristics of motion as observed in nature.

In the corollaries to his laws Newton tries to show how the composition and resolution of forces by the parallelogram rule follows from his definition. In deriving this result he tacitly assumes that the action of any force on a particle takes place independently of the action of any other forces that may be acting on the particle at the same time, a principle that would seem to deserve explicit statement. Some writers on mechanics prefer to replace Newton's second law by this *principle of the independence of the action of forces.*

187. The third law expresses the physical fact that in nature all forces occur in pairs of equal and opposite forces. Two such equal and opposite forces in the same line are often said to constitute a *stress*. Newton's third law has therefore been called the **law of stress.**

This law, which was first clearly conceived in Newton's time, involves what may be regarded as the second fundamental property of matter or mass (the first being its indestructibility); viz. that *any two particles of matter determine in each other oppositely directed accelerations along the line joining them.*

CHAPTER X.

STATICS OF THE PARTICLE.

188. According to the definition of force (Arts. 171, 173), a single force F acting on a particle of mass m produces an acceleration j such that $F = mj$; *i. e.* the vector F is m times the vector j.

If two forces F_1, F_2 act on the same particle, it is assumed (Art. 186) that each acts as if the other were not present; hence, if j_1, j_2 are the accelerations which F_1, F_2 would produce separately, *i. e.* if $F_1 = mj_1$, $F_2 = mj_2$, then the combined effect of F_1 and F_2 will be to produce an acceleration equal to the resultant, or geometric sum, $j = j_1 + j_2$, of the accelerations j_1, j_2; and this resultant acceleration j can be produced by a single force $R = mj$ (Fig. 41).

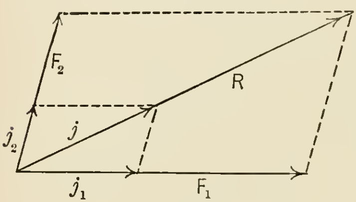

Fig. 41.

The combined effect of the two forces F_1, F_2 acting on the same particle m is thus the same as that of that single force R which is the resultant, or geometric sum, of F_1 and F_2. The two forces F_1, F_2 are said to be **equivalent** to the single force R; R is called the **resultant** of F_1, F_2, which are called **components** of R.

189. Thus, the resultant R of two forces F_1, F_2 acting on the same particle is found (Fig. 42) as the diagonal of the parallelogram constructed with F_1, F_2 as adjacent sides.

Hence

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta},$$

$$\frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha} = \frac{R}{\sin\theta},$$

where θ is the angle between F_1 and F_2, α that between R and F_1, β that between R and F_2.

This proposition is known as the **parallelogram of forces.** It enables us to find the vector R when the vectors F_1, F_2 are given; and conversely, to find F_1, F_2 if, in addition to the

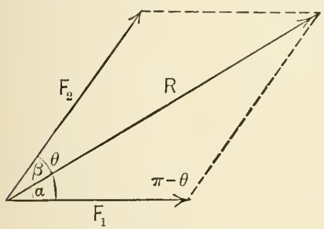

Fig. 42.

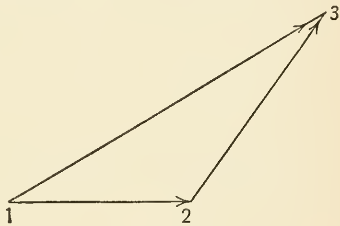

Fig. 43.

vector R, the directions of F_1, F_2 (the angles α, β) are given. The latter operation is called **resolving** a force along given directions.

To find R when F_1, F_2 are given it suffices (instead of constructing the whole parallelogram) to lay off (Fig. 43) 1 2, equal to the vector F_1 (in magnitude, direction, and sense), and 2 3, equal to the vector F_2; then 1 3 is the resultant R. 123 is called the *triangle of forces.*

190. Let any number n of forces F_1, F_2, $\cdots$ F_n be applied at the same point O, *i. e.* act on the same particle at O. By Art. 189, we can find the resultant R_1 of F_1 and F_2, next the resultant R_2 of R_1 and F_3, then the resultant R_3 of R_2 and F_4, and so on. The resultant R of R_{n-2} and F_n is evidently equivalent to the whole system F_1, F_2, F_3, $\cdots$ F_n, and is

called its **resultant.** It thus appears that *a system consisting of any number of forces acting on the same particle is equivalent to a single resultant.*

It may of course happen that this resultant is zero. In this case the system is said to be **in equilibrium.** *The condition of equilibrium of a system of forces acting on the same particle* is therefore:

$$R = 0.$$

The system of forces in this case produces no acceleration; notice that equilibrium of the forces does not mean that the particle is at rest. Under forces that are in equilibrium the particle, if at rest, will remain at rest; if in motion, it will continue to move uniformly in a straight line.

191. In practice, the process of finding the resultant indicated in Art. 190 is inconvenient when the number of forces is large. If the forces are given geometrically, as

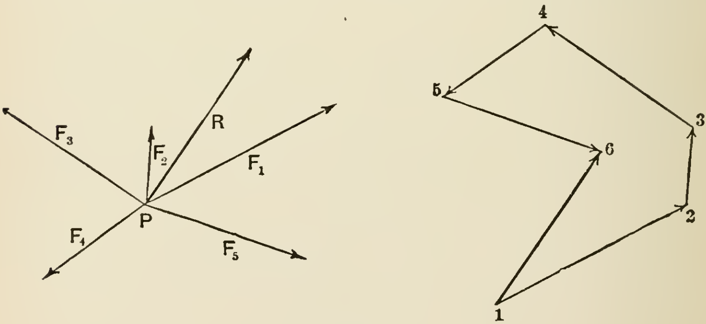

Fig. 44.

vectors, we have only to add these vectors; and this can best be done in a separate diagram, called the **force polygon.** Thus, in Fig. 44, 1 2 is drawn equal and parallel to F_1, 2 3 equal and parallel to F_2, 3 4 to F_3, 4 5 to F_4, 5 6 to F_5. The

closing line of the force polygon, viz. 1 6 in the figure, is equal and parallel to the resultant R, which is therefore obtained by drawing through the point of application of the forces a line equal and parallel to 1 6.

The geometrical condition of equilibrium consists in the closing of the force polygon, that is, in the coincidence of its terminal point 6 with its initial point 1.

192. Analytically, *a system of concurrent forces is reduced to its most simple equivalent form, i. e.* to its single resultant, by resolving each force F into three components X, Y, Z, along three rectangular axes passing through the particle, or point of application of the given forces. All components lying in the direction of the same axis can then be added algebraically, and the whole system of forces is found to be equivalent to three rectangular forces ΣX, ΣY, ΣZ, which, by the parallelogram law, can be replaced by a single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

The angles α, β, γ made by this resultant with the axes are given by the relations

$$\frac{\cos\alpha}{\Sigma X} = \frac{\cos\beta}{\Sigma Y} = \frac{\cos\gamma}{\Sigma Z} = \frac{1}{R}.$$

193. If the forces all lie in the same plane, only two axes are required and we have

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}, \quad \tan\theta = \frac{\Sigma Y}{\Sigma X},$$

where θ is the angle between the axis of X and R.

194. The condition of equilibrium (Art. 190) $R = 0$ becomes, by Art. 192, $(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0$. As all terms in the left-hand member are positive, their sum can vanish only when each term is zero. *The analytical conditions*

of the equilibrium of any number of concurrent forces are therefore:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

195. The forces of nature receive various special names according to the circumstances under which they occur. Thus the **weight** of a mass has already been defined (Art. 176), as the force with which the mass is attracted by the mass of the earth.

A string carrying a mass at one end and suspended from a fixed point, is subjected to a certain tension. This means that if the string were cut it would require the application of a force along the line of the string to keep the weight in equilibrium. This force, which may thus serve to replace the action of the string, is called its **tension.**

When the surfaces of two physical bodies A, B are in contact, a **pressure** may exist between them; that is, if one of the bodies, say B, be removed, it may require the introduction of a force to keep A in the same state of rest or motion that it had before the removal of B. This force, which acts along the common normal of the surfaces at the point of contact if there is no friction, is called the **resistance,** or **reaction,** of B, and a force equal and opposite to it is called the **pressure** exerted by A on B. For the case of friction see Arts. 237 sq.

[**196. Exercises.**

(1) Show that the resultant of two equal forces F including an angle θ is $2F\cos\frac{1}{2}\theta$. Observe the variation of the resultant as θ varies from 0 to π; for what angle θ is the resultant equal to F?

(2) Show that the resultant of two forces OA, OB is twice OC, where C is the midpoint of A and B.

(3) Find the magnitude and direction of the resultant of two forces of 12 and 20 lb., including an angle of 60°.

(4) Find the resultant of 6 equal concurrent forces, each inclined to the next at 45°.

(5) Show that the forces OA, OB, OC are in equilibrium if O is the centroid of the triangular area ABC.

(6) Show (by Art. 194) that if any number of concurrent forces are in equilibrium, their point of concurrence is the centroid of their extremities.

(7) A mass m rests on a plane inclined to the horizon at an angle θ; it is kept in equilibrium: (a) by a force P_1 parallel to the plane; (b) by a horizontal force P_2; (c) by a force P_3 inclined to the horizon at an angle $\theta + \alpha$. Determine in each case the force P and the pressure R on the plane.

(8) A weight W is suspended from two fixed points A, B by means of a string ACB, C being the point of the string where the weight W is attached. If AC, BC be inclined to the vertical at angles α, β, find the tensions in AC, BC: (a) analytically; (b) graphically.

(9) Show that the resultant R of three concurrent forces A, B, C in the same plane is given by $R^2 = A^2 + B^2 + C^2 + 2BC\cos(B, C) + 2CA\cos(C, A) + 2AB\cos(A, B)$.

(10) A weightless rod AC, hinged at one end A so as to be free to turn in a vertical plane, is held in a horizontal position by means of the chain BC, the point B lying vertically above A. If a weight W be suspended at C, find the thrust P in AC and the tension T of the chain. Assume $AC = 8$ ft., $AB = 6$ ft.

(11) In Ex. (10), suppose the rod AC, instead of being hinged at A, to be set firmly into the wall in a horizontal position; and let the chain fastened at B run at C over a smooth pulley and carry the weight W. Find the tension of the chain and the magnitude and direction of the pressure on the pulley at C.

(12) In "tacking against the wind," let W be the force of the wind; α, β the angles made by the axis of the boat with the direction in which the wind blows, and with the sail, respectively. Determine the force that drives the boat forward and find for what position of the sail it is greatest.

(13) A cylinder of weight W rests on two inclined planes whose intersection is horizontal and parallel to the axis of the cylinder. Find the pressures on these planes.

(14) Find the tensions in the string $ABCD$, fixed at A and D, and carrying equal weights W at B and C, if $AD = c$ is horizontal, $AB = BC = CD$, and the length of the string is $3l$.

(15) In the toggle-joint press two equal rods CA, CB are hinged at C; a force F bisecting the angle 2α between the rods forces the ends A, B apart. If A be fixed, find the pressure exerted at B at right angles to F if $F = 100$ lbs. and $\alpha = 15°, 30°, 45°, 60°, 75°, 90°$.

(16) A stone weighing 800 lbs. hangs from a derrick by a chain 15 ft. long. If pulled 5 ft. away from the vertical by means of a horizontal rope attached to it, what are the tensions of the chain and the rope? What if pulled 9 ft. away?

(17) A rope 16 ft. long has its ends fastened to two points, 10 ft. apart, at the same height above the ground; a weight W is suspended from the rope by means of a ring free to slide along the rope. Find the tension of the rope.

(18) A string with equal weights W attached to its ends is hung over two smooth pegs A, B fixed in a vertical wall. Find the pressure on the pegs: (a) when the line AB is horizontal; (b) when it is inclined to the horizon at an angle θ.

CHAPTER XI.

STATICS OF THE RIGID BODY.

197. A system of forces acting on a rigid body can, in general, not be reduced to a single resultant, as is the case for concurrent forces (Art. 190); in other words, there does not always exist a single force having the same effect that the system of forces has in changing the motion of the body.

Before discussing the general case it is best to consider certain particular kinds of systems of forces, viz. *concurrent*, *parallel*, and *complanar* systems.

Throughout the statics of the rigid body it is assumed that *the effect of a force is not changed if the force is transferred to any other position on its line of action;* in other words, a body is called *rigid* if, and only if, it possesses this property. Thus the "point of application" of a force acting on a rigid body is not an essential characteristic of the force; what characterizes the force is its magnitude, line of action, and sense. This is what is meant by saying that a force is a *localized vector* or *rotor* (Art. 182).

1. Concurrent forces.

198. In the case of concurrent forces there exists a single resultant, viz. the geometric sum of the forces. If this resultant happens to be zero, *i. e.* if the force polygon (Art. 191) closes, the forces are in equilibrium.

As the projection of a closed polygon on any line is zero, it follows that *the projection of the resultant on any line is equal to the algebraic sum of the projections of its components.*

Thus, if the forces P, Q intersect at O and have the resultant R we find by projecting on any line l:

$$R \cos(l, R) = P \cos(l, P) + Q \cos(l, Q).$$

Let the line l be drawn through O, in the plane of P and Q, and let an arbitrary length $OS = s$ (Fig. 45) be laid off at right angles to l in the same plane. Then, multiplying the last equation by s we find

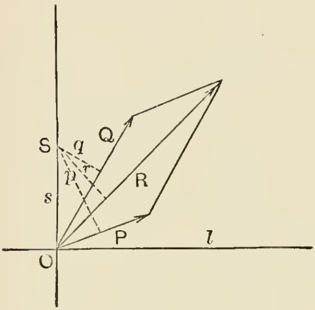

Fig. 45.

$$R \cdot s \cos(l, R) = P \cdot s \cos(l, P) + Q \cdot s \cos(l, Q);$$

or since $s \cos(l, R) = r$, $s \cos(l, P) = p$, $s \cos(l, Q) = q$ are the perpendiculars from S to R, P, Q:

$$Rr = Pp + Qq.$$

Now the product of a force into its perpendicular distance from a point is called the **moment** of the force about the point; the product is taken with the positive or negative sign according as the force tends to turn counterclockwise or clockwise about the point. We have therefore proved that *the algebraic sum of the moments of any two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point.*

This proposition is known as the **theorem of moments,** or Varignon's theorem. It is readily extended to any number of concurrent forces in the same plane. As a corollary it follows that *the sum of the moments of any such forces about any point of their resultant is zero.*

199. As the moment of a force represents twice the area of the triangle having the force as base and the reference

point as vertex, the theorem of moments can also be proved by comparing areas. Thus, with the notation of Fig. 46 we have

$$SOR = SOQ + SQR + QOR,$$

i. e.

$$R \cdot r = Q \cdot q + P \cdot ST + P \cdot TU,$$

or since

$$ST + TU = SU = p:$$

$$Rr = Qq + Pp.$$

It is often convenient to think of the moment Rr of a force R about the point S as a vector drawn through S at right angles to the plane determined by S and R. This is in agreement with the representation of a parallelogram area by such a vector, mentioned in Art. 119. Indeed, the moment Rr is the cross-product of the radius vector drawn from S to any point of R into the force-vector R.

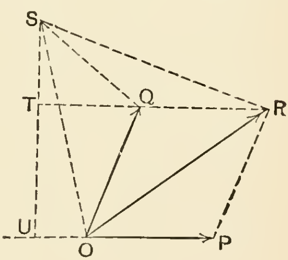

Fig. 46.

This representation is of special advantage when the concurrent forces do not lie in the same plane. It can then be shown that the moment of the resultant about any point is equal to the geometric sum of the vectors representing the moments of the components.

2. Parallel forces.

200. It will be proved in the next article that any two parallel forces acting on a rigid body have a single resultant, except when the two parallel forces are of equal magnitude and opposite sense. In the latter case, the two equal and opposite parallel forces are said to constitute a **couple,** and no further reduction is possible.

It follows readily that *any system of parallel forces acting on a rigid body can be reduced either to a single force or to a single couple.*

201. Resultant of two parallel forces. In the plane of the given parallel forces P, Q, resolve P, at any point p of its line of action, into any two components, say P' and F (Fig. 47); and at the point q where F meets the line of Q,

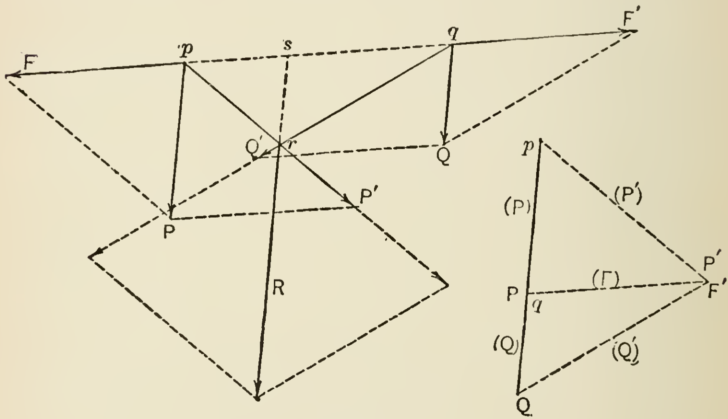

Fig. 47.

resolve Q into two components F', Q', selecting for F' a force equal and opposite to, and in the same line with, F. The two equal and opposite forces F, F' in the same line pq have no effect on the rigid body so that the given forces P, Q are together equivalent to the two components P', Q' alone. The lines of P' and Q' will in general intersect at a point r and these forces can therefore be replaced by a resultant R passing through r.

By placing the triangles $pP'P$ and $qF'Q$ together so that their equal sides PP' and qF' coincide (as is done in Fig. 47, on the right) it appears at once that the resultant of P' and

Q', and hence *the resultant R of P and Q*, is *parallel to P and Q and in magnitude equal to the algebraic sum of P and Q:*

$$R = P + Q.$$

In Fig. 47, the two given parallel forces P, Q were assumed of the same sense. The construction applies, however, equally well to the case when they are of opposite sense. The resultant R will then be found to lie not between P and Q, but outside, on the side of the larger force. The construction fails only when the two given forces are equal and of opposite sense, since then the lines pP' and qQ' become parallel. This exceptional case will be considered in Art. 208.

202. The theorem of moments for parallel forces. As the forces R, P', Q' (Fig. 47) are concurrent the theorem of moments (Art. 198) can be applied to these three forces. Hence, taking moments about any point S of the plane of P' and Q' and denoting the perpendiculars from S to the forces by the corresponding small letters, we have·

$$Rr = P'p' + Q'q'.$$

Now P' can be regarded as the resultant of P and $-F$, and Q' as the resultant of Q and $-F'$; hence

$$P'p' = Pp - Ff, \quad Q'q' = Qq - F'f';$$

substituting these values and remembering that F and F' are equal and opposite and in the same line, we find

$$Rr = Pp + Qq;$$

i. e. the sum of the moments of two parallel forces about any point in their plane is equal to the moment of their resultant about the same point.

If, in particular, the point of reference be taken on the resultant so that $r = 0$, we find

$$Pp = -Qq;$$

i. e. the resultant of two parallel forces divides their distance in the inverse ratio of the forces.

This proposition, well known from its application to the lever, is often referred to as the *principle of the lever.*

203. It has been shown that two parallel forces P, Q acting on a rigid body, provided they are not equal and of opposite sense, have a resultant $R = P + Q$, parallel to P and Q, and that its position in the rigid body can be found either analytically from the fact that R divides the distance between P and Q in the inverse ratio of these forces, or geometrically by the construction of Art. 201.

This *geometrical construction* is best carried out in the following order (Fig. 48). The parallel forces P, Q being

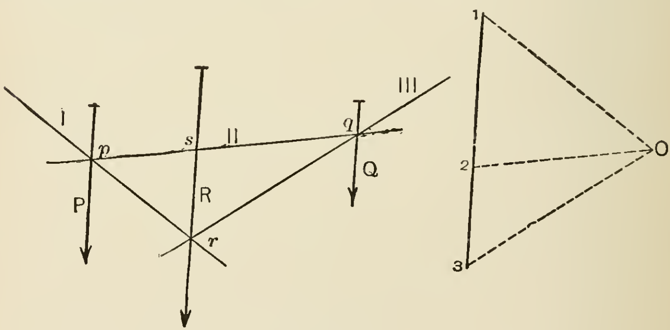

Fig. 48.

given in position, begin by constructing the **force polygon,** which here consists merely of a straight line on which the forces $P = 1\,2$, $Q = 2\,3$ are laid off to scale; the closing line, $1\,3$, gives the resultant in magnitude, direction, and sense; it only remains to find its position, and for this it suffices to find one point of its line of action.

Now, to resolve P and Q each into two components (as is done in Art. 201) so that one component of P and

one of Q are equal and opposite and in the same line, it is only necessary to draw from an arbitrary point O, called the *pole*, the lines $O\,1$, $O\,2$, $O\,3$; then $1\,O$, $O\,2$ can be regarded as components of $P = 1\,2$, and $2\,O$, $O\,3$ as components of $Q = 2\,3$. Next construct the so-called **funicular polygon** by drawing a line I parallel to $O\,1$, intersecting P say at p; through p a line II parallel to $O\,2$ meeting Q say at q; through q a line III parallel to $O\,3$.

The intersection r of I and III is a point of the resultant R as appears by comparing Figs. 48 and 47; Fig. 48 being the same as Fig. 47, with the superfluous lines left out.

204. Analytically, the resultant of n parallel forces F_1, $F_2, \cdots F_n$, whether in the same plane or not, can be found as follows:

The resultant of F_1 and F_2 is a force $F_1 + F_2$ situated in the plane (F_1, F_2), so that $F_1p_1 = F_2p_2$ (Art. 202), where p_1, p_2 are the (perpendicular or oblique) distances of the resultant from F_1 and F_2, respectively. This resultant $F_1 + F_2$ can now be combined with F_3 to form a resultant $F_1 + F_2 + F_3$, whose distances from $F_1 + F_2$ and F_3 in the plane determined by these two forces are as F_3 is to $F_1 + F_2$. This process can be continued until all forces have been combined; the final resultant is

$$F_1 + F_2 + \cdots + F_n.$$

Any number of parallel forces are, therefore, *equivalent to a single resultant equal to their algebraic sum*, provided this sum does not vanish.

205. To find the *position* of this resultant analytically, let the points of application of the forces F_1, $F_2, \cdots F_n$ be (x_1, y_1, z_1), (x_2, y_2, z_2), $\cdots$ (x_n, y_n, z_n). The point of application of the resultant $F_1 + F_2$ of F_1 and F_2 may be taken so as

to divide the distance of the points of application of F_1 and F_2 in the ratio F_2/F_1; hence, denoting its co-ordinates by x', y', z', we have $F_1(x' - x_1) = F_2(x_2 - x')$, or

$$(F_1 + F_2)x' = F_1x_1 + F_2x_2,$$

and similarly for y' and z'.

The force $F_1 + F_2$ combines with F_3 to form a resultant $F_1+F_2+F_3$, whose point of application (x'',y'', z'') is given by

$$(F_1 + F_2 + F_3)x'' = F_1x_1 + F_2x_2 + F_3x_3,$$

with similar expressions for y'', z''.

Proceeding in this way, we find for the point of application $(\bar{x}, \bar{y}, \bar{z})$ of the resultant of all the given forces

$$(F_1 + F_2 + \cdots + F_n)\bar{x} = F_1x_1 + F_2x_2 + \cdots + F_nx_n,$$

with corresponding equations for $\bar{y}$ and $\bar{z}$. We may write these equations in the form:

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}, \quad \bar{y} = \frac{\Sigma Fy}{\Sigma F}, \quad \bar{z} = \frac{\Sigma Fz}{\Sigma F},$$

unless $\Sigma F = 0$.

As these expressions for $\bar{x}$, $\bar{y}$, $\bar{z}$ are independent of the direction of the parallel forces it follows that the same point $(\bar{x}, \bar{y}, \bar{z})$ would be found if the forces were all turned in any way about their points of application, provided they remain parallel. The point $(\bar{x}, \bar{y}, \bar{z})$ is for this reason called the **center** of the system of parallel forces. It is nothing but the centroid of the points of application if these points are regarded as possessing masses equal to the magnitudes of the forces.

206. Conditions of equilibrium. It follows from what precedes that *for the equilibrium of a system of parallel forces the condition* $\Sigma F = 0$, *or* $R = 0$, *though always necessary, is not sufficient.*

Now, if the resultant R of the n parallel forces $F_1, F_2, \cdots F_n$ is zero, the resultant R' of the $n-1$ forces $F_1, F_2, \cdots F_{n-1}$ cannot be zero, and its point of application is found (by Art. 205) from $\bar{x} = (F_1x_1 + F_2x_2 + \cdots + F_{n-1}x_{n-1})/(F_1 + F_2 + \cdots F_{n-1})$ and similar expressions for $\bar{y}$ and $\bar{z}$. The whole system of parallel forces is therefore equivalent to the two parallel forces R' and F_n. Two such forces can be in equilibrium only when they lie in the same straight line; *i. e.* F_n must lie in the same line with R' and must therefore pass through the point $(\bar{x}, \bar{y}, \bar{z})$, which is a point of R'.

The additional condition of equilibrium is, therefore,

$$\frac{\bar{x} - x_n}{\cos\alpha} = \frac{\bar{y} - y_n}{\cos\beta} = \frac{\bar{z} - z_n}{\cos\gamma},$$

where α, β, γ are the angles made by the direction of the forces with the axes.

For practical application it is usually best to replace the last condition by taking moments about a convenient point. Thus, the analytical conditions of equilibrium can be written in the form

$$\Sigma F = 0, \quad \Sigma Fp = 0.$$

Graphically, to the former corresponds the closing of the force-polygon, to the latter, in the case of complanar forces, the closing of the funicular polygon.

207. Weight; center of gravity. The most important special case of parallel forces is that of the force of gravity which acts at any given place near the earth's surface in approximately parallel lines on every particle of matter.

If g be the acceleration of gravity, the force of gravity on a particle of mass m is

$$w = mg,$$

and is called the **weight** of the particle or of the mass m.

For a system of particles of masses $m_1, m_2, \cdots m_n$ we have

$$w_1 = m_1 g, \quad w_2 = m_2 g, \quad \cdots w_n = m_n g.$$

If the particles are rigidly connected, the resultant W of these parallel forces,

$$W = w_1 + w_2 + \cdots + w_n = (m_1 + m_2 + \cdots + m_n)g = Mg,$$

where M is the mass of the system, is called the weight of the system.

The center of the parallel forces of gravity of a system of rigidly connected particles has, by Art. 205, the co-ordinates

$$\bar{x} = \frac{\Sigma mgx}{\Sigma mg}, \quad \bar{y} = \frac{\Sigma mgy}{\Sigma mg}, \quad \bar{z} = \frac{\Sigma mgz}{\Sigma mg},$$

or since the constant g cancels,

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}.$$

This point is called the **center of gravity** of the system, and is evidently identical with the *center of mass*, or **centroid** (see Art. 159).

For continuous masses the same formulæ hold, except that the summations become integrations.

The *weight* W of a physical body of mass M is therefore a vertical force passing through the centroid of its mass.

3. Theory of couples.

208. The construction given for the resultant of two parallel forces given in Arts. 201 and 203 fails if, and only if, the given forces are equal and of opposite sense. In this case, the lines pP' and qQ' in Fig. 47, and the lines I and III of the funicular polygon (Fig. 48), become parallel, so that their intersection r lies at infinity. The magnitude of the resultant is of course zero.

The combination of two equal and opposite parallel forces $(F, -F)$ acting on a rigid body is called a **couple**. *A couple is, therefore, not equivalent to a single force,* although it might be said to be equivalent to the limit of a force whose magnitude approaches zero while its line of action is removed to infinity.

The perpendicular distance $AB = p$ (Fig. 49) of the forces of the couple is called the **arm**, and the product Fp of the force F into the arm p is called the **moment** of the couple. The moment, or the couple itself, is also called a **torque**.

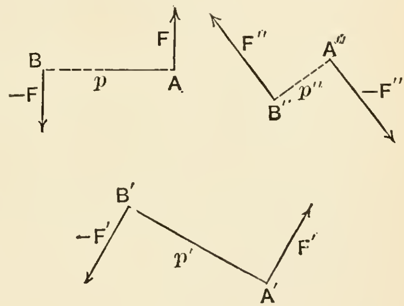

Fig. 49.

Notice that *the moment of a couple is simply the sum of the moments of its forces about any point in its plane.*

If we imagine the couple (F, p) to act upon an invariable plane figure in its plane, and if the midpoint of its arm be a fixed point of this figure, the couple will evidently tend to turn the figure about this midpoint. (It is to be observed that it is *not* true, in general, that a couple acting on a rigid body produces rotation about an axis at right angles to its plane.) A couple of the type (F, p) or (F', p') (see Fig. 49) will tend to rotate counterclockwise, while a couple of the type (F'', p'') tends to turn clockwise. Couples in the same plane, or in parallel planes, are therefore distinguished as to their **sense** and this sense is expressed by the algebraic sign attributed to the moment. Thus, the moment of the couple (F, p) in Fig. 49 is $+ Fp$, that of the couple (F'', p'') is $- F''p''$.

209. *The effect of a couple is not changed by translation, i. e.* by moving its plane parallel to itself without rotating it.

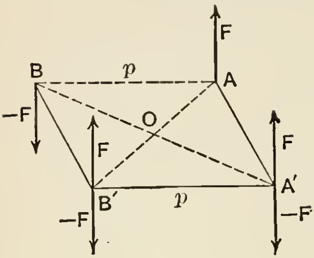

Fig. 50.

Let $AB = p$ (Fig. 50) be the arm of the couple (F, p) in its original position, and $A'B'$ the same arm in a new position parallel to the original one in the same plane, or in any parallel plane. By introducing at each end of the new arm $A'B'$ two opposite forces F, $-F$, each equal and parallel to the original forces F, the given system is not changed. But the two equal and parallel forces F at A and B' form a resultant $2F$ at the midpoint O of the diagonal AB' of the parallelogram $ABB'A'$. Similarly, the two forces $-F$ at B and A' are together equivalent to a resultant $-2F$ at the same point O. These two resultant forces, being equal and opposite and acting in the same line, are together equivalent to zero. Hence the whole system reduces to the force F at A' and the force $-F$ at B', which form, therefore, a couple equivalent to the original couple at AB.

210. *The effect of a couple is not changed by rotation in its plane.*

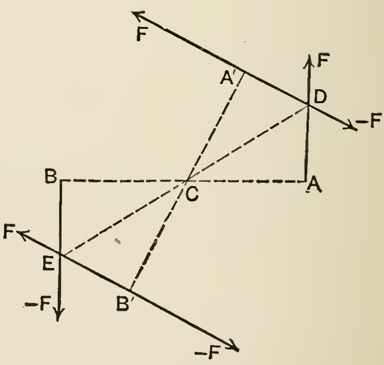

Fig. 51.

Let AB (Fig 51) be the arm of the couple in the original position, C its midpoint, and let the arm be turned about C into the position $A'B'$. Applying again at A', B' equal and opposite forces each equal to F, the forces $-F$ at A' and F at A will form a resultant acting along CD, while F at B' and $-F$ at B give an equal and opposite resultant along CE. These two resultant forces destroy each other and leave nothing but the couple formed by F at A' and $-F$ at B' which is therefore equivalent to the original couple.

Any other displacement of the couple in its plane, or to a parallel plane, can be effected by a translation combined with a rotation in its plane about the midpoint of its arm. *The effect of a couple is therefore not changed by any displacement in its plane or to a parallel plane.*

211. *The effect of a couple is not changed if its force F and its arm p be changed simultaneously in any way, provided their product Fp remain the same.*

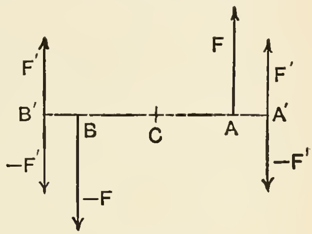

Fig. 52.

Let $AB = p$ be the original arm (Fig. 52), F the original force of the couple; and let $A'B' = p'$ be the new arm. The introduction of two equal and opposite forces F' at A', and also at B', will not change the given system F, $-F$. Now, selecting for F' a magnitude such that $F'p' = Fp$, the force F at A and the force $-F'$ at A' combine (Arts. 201, 203) to form a parallel resultant through C, the midpoint of the arm, since for this point $F \cdot \frac{1}{2}p + (-F') \cdot \frac{1}{2}p' = 0$. Similarly, $-F$ at B and F' at B' give a resultant of the same magnitude, in the same line through C, but of opposite sense. These two resultant forces thus destroying each other, there remains only the couple formed by F' at A' and $-F'$ at B', for which $Fp = F'p'$.

212. It results from the last three articles that the only essential characteristics of a couple are: (*a*) the numerical value of the moment; (*b*) the sense, or direction of rotation; and (*c*) what has been called the "aspect" of its plane, *i. e.* the direction of any normal to this plane.

It is to be noticed that the plane of the two forces forming the couple is not an essential characteristic of the couple; just as the point of application of a force is not an essential characteristic of the force (see Art. 197); provided, of course, that the couple (or force) is acting on a *rigid* body.

Now the three characteristics enumerated above can all be indicated by a *vector* which can therefore serve as the geometrical representative of the couple. Thus, the couple formed by the forces F, $-F$ (Fig. 53), whose perpendicular distance is p, is represented by the vector $AB = Fp$ laid off on any normal to the plane of the couple.

12

The sense is indicated by drawing the vector toward that side of the plane from which the couple is seen to rotate counterclockwise.

We shall call this geometrical representative AB of the couple simply the **vector** of the couple. It is sometimes called its *moment*, or its *axis*, or its *axial moment*.

213. As was pointed out in Art. 208, a couple can be regarded as the limit of a force whose magnitude approaches zero while its line of action is removed to infinity. Similarly, in kinematics an angular velocity whose magnitude tends to zero while its axis is removed indefinitely becomes in the limit a *velocity of translation*.

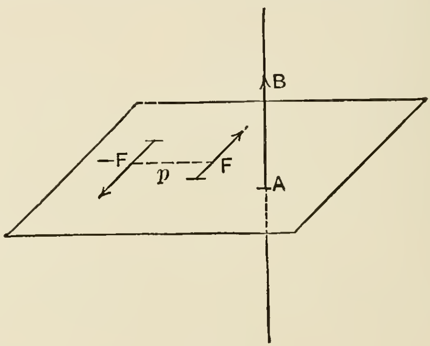

Fig. 53.

Just as, in kinematics (see Art. 122), two equal and opposite angular velocities about parallel axes produce a velocity of translation, so in statics two equal and opposite forces along parallel lines form a new kind of quantity called a *couple*.

It should, however, be noticed that while angular velocities and forces are represented by *rotors*, *i. e.* by vectors confined to definite lines, velocities of translation and couples have for their geometrical representatives vectors not confined to particular lines.

It is due to this analogy between the two fundamental conceptions that a certain dualism exists between the theories of statics and kinematics, so that a large portion of the theory of kinematics of a rigid body might be made directly available for statics by simply substituting for angular velocity and velocity of translation the corresponding ideas of force and couple.

214. It is easily seen how, by means of Arts. 209–211, any number of couples acting on a rigid body can be reduced to a single resultant couple. It can also be proved without much difficulty that the vector of the resultant couple is the geometric sum of the vectors of the given couples; in other words, *vectors representing couples acting on the same rigid body are combined by the parallelogram law.*

In the particular case when the couples all lie in parallel planes, or in the same plane, their vectors may be taken in the same line and can, therefore, be added algebraically. Generally, *the resultant of any number of couples is a single couple whose vector is the geometric sum of the vectors of the given couples.*

Conversely, a couple can be resolved into components by resolving its vector into components.

215. To combine a single force P with a couple (F,p) lying in the same plane it is only necessary to place the couple in its plane in such a position (Fig. 54) that one of its forces, say $-F$, shall lie in the same line and in opposite sense with the single force P, and to transform the couple (F, p) into a couple (P, p'), by Art. 211, so that $Fp = Pp'$. The original force P and the force $-P$ of the transformed couple destroying each other at A, there remains only the other force P, at A', of the transformed couple, that is, a force parallel and equal to the original single force P, at the distance

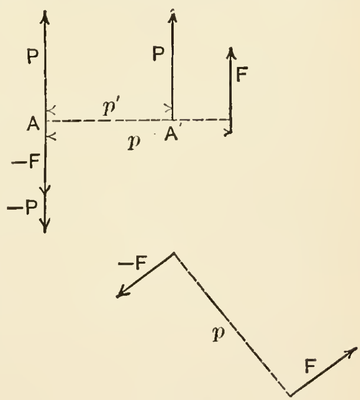

Fig. 54.

$$p' = \frac{F}{P} p$$

from it.

Hence, *a couple and a single force in the same plane are together equivalent to a single force equal and parallel to, and of the same sense with, the given force, but at a distance from it which is found by dividing the moment of the couple by the single force.*

Conversely, *a single force P applied at a point A of a rigid body can always be replaced by an equal and parallel force P of the same sense, applied at any other point A' of the same body, in combination with the couple formed by P at A and $-P$ at A'.*

This follows at once by applying at A' two equal and opposite forces each equal and parallel to P.

216. The proposition of Art. 215 applies even when the force lies in a plane parallel to that of the couple, since the couple can be transferred to any parallel plane without changing its effect.

If the single force intersects the plane of the couple, it can be resolved into two components, one lying in the plane of the couple, while the other is at right angles to this plane. On the former component the couple has, according to Art. 215, the effect of transferring it to a parallel line. We thus obtain *two non-intersecting*, or *skew, forces at right angles to each other.*

Let P be the given force, and let it make the angle α with the plane of the given couple, whose force is F and whose arm is p. Then $P \sin\alpha$ is the component at right angles to the plane of the couple, while $P \cos\alpha$ combined with the couple whose moment is Fp is equivalent to a force $P \cos\alpha$ in the plane of the couple; this force $P \cos\alpha$ is parallel to the pro-

jection of P on the plane, and has the distance $Fp/P \cos\alpha$ from this projection.

Hence, in the most general case, *the combination of a single force and a couple can be replaced by the combination of two single forces crossing each other (without meeting) at right angles;* it can be reduced to a single force only when the force is parallel to the plane of the couple.

4. Complanar forces.

217. If the forces acting on a rigid body all lie in the same plane, *i. e.* if the forces are **complanar**, the system can be reduced to a single force and a single couple by applying the last proposition of Art. 215. For, selecting an arbitrary point O of the plane as point of reference, we can replace each force F of the system by an equal force F applied at O, together with a couple Fp, whose arm p is the perpendicular from O to the line of action of the given force F at P.

We thus obtain, in the plane, a number of concurrent forces at O which are equivalent to a *single resultant* R, passing through O and equal to the geometric sum of the given forces; and in addition a number of couples in the same plane which give a *single resultant couple*, say $H = \Sigma Fp$.

Notice that the moment H of the resultant couple is simply the sum of the moments about O of all the given forces.

It follows that the **conditions of equilibrium** are:

$$R = 0, \quad H = 0;$$

i. e. a system of complanar forces is in equilibrium if, and only if, (a) its resultant is zero, and (b) the algebraic sum of the moments of all its forces is zero about any point in its plane.

218. By Art. 217, a system of complanar forces reduces, for any point of reference O in its plane, to a force R and a

couple H. But as these lie in the same plane, it follows from the first proposition of Art. 215 that they can be reduced to a single resultant R (unless $R = 0$). The distance r of this single resultant from O is such that $Rr = -H$; *i. e.* $r = -H/R$. The line of action of this single resultant is called the *central axis* of the system.

Thus, a system of complanar forces can always be reduced either to a single force R or to a single couple H.

219. For a purely analytical reduction of a plane system of forces the system is referred to rectangular axes Ox, Oy, arbitrarily assumed in the plane (Fig. 55). Every force F is

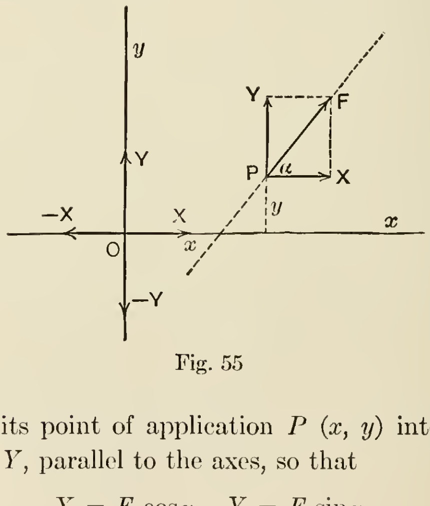

Fig. 55

resolved at its point of application P (x, y) into two components X, Y, parallel to the axes, so that

$$X = F \cos\alpha, \quad Y = F \sin\alpha,$$

α being the angle made by F with the axis Ox. At the origin O two equal and opposite forces X, $-X$ are applied along Ox, and two equal and opposite forces Y, $-Y$ along Oy. Thus, X at P is equivalent to X at O together with the couple formed by X at P and $-X$ at O; the moment of

this couple is evidently $-yX$. Similarly, Y at P is replaced by Y at O together with a couple whose moment is xY. The force F at P is therefore equivalent to the two forces X, Y at O together with a couple whose moment is $xY - yX$.

Proceeding in the same way with every given force, we obtain a number of forces X along Ox whose algebraic sum we call ΣX, and a number of forces Y along Oy which give ΣY. These two rectangular forces form the resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

whose direction is given by

$$\tan\alpha = \frac{\Sigma Y}{\Sigma X},$$

where α is the angle between Ox and R.

In addition to this, we obtain a number of couples $xY - yX$ whose algebraic sum forms the resulting couple

$$H = \Sigma(xY - yX).$$

The whole system is thus found equivalent to a resultant force R together with a resultant couple H in the same plane with R The *conditions of equilibrium* $R = 0$, $H = 0$ (Art. 217) can therefore be expressed analytically by the three equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma(xY - yX) = 0.$$

220. If R be not zero, R and H can be reduced to a single resultant R' equal and parallel to R at the distance $-H/R$ from it (see Art. 218). The equation of the line of this single resultant R', *i. e.* the *central axis* of the system of forces, is found by considering that it makes the angle α with the axis of x and that its distance from the origin is

$$H/R = \Sigma(xY - yX)/\sqrt{(\Sigma X)^2 + (\Sigma Y)^2}.$$

Hence its equation is

$$\xi \cdot \Sigma Y - \eta \cdot \Sigma X - \Sigma(xY - yX) = 0.$$

If $R = 0$, the system is equivalent to the couple

$$H = \Sigma(xY - yX).$$

If H itself be also zero, the system is in equilibrium.

221. Exercises.

(1) *A homogeneous straight rod* $AB = 2l$ (Fig. 56) *of weight* W *rests with one end* A *on a smooth horizontal plane* AH, *and with the point* $E(AE = e)$ *on a cylindrical support, the axis of the cylinder being at right angles to the vertical plane containing the rod. Determine what*

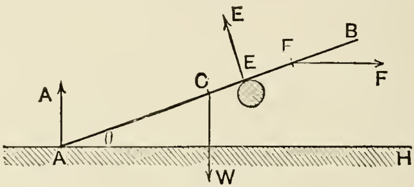

Fig. 56.

horizontal force F *must be applied at a given point* F *of the rod* $(AF = f > e)$ *to keep the rod in equilibrium when inclined to the horizon at an angle* θ.

The rod exerts a certain unknown pressure on each of the supports at A and E, in the direction of the normals to the surfaces of contact, provided there be no friction, as is here assumed. The supports may therefore be imagined removed if forces A, E, equal and opposite to these pressures, be introduced; these forces A, E are called the *reactions* of the supports. The rod itself is here regarded as a straight line; its weight W is applied at its middle point C.

Taking A as origin and AH as axis of x, the resolution of the forces gives

$$\Sigma X \equiv F - E \sin\theta = 0, \tag{1}$$

$$\Sigma Y \equiv A - W + E \cos\theta = 0. \tag{2}$$

Taking moments about A, we find

$$E \cdot e - W \cdot l \cos\theta - F \cdot f \sin\theta = 0. \tag{3}$$

Eliminating F from (1) and (3), we have

$$E = \frac{l \cos\theta}{e - f \sin^2\theta} W,$$

hence from (2),

$$A = \left(1 - \frac{l \cos^2\theta}{e - f \sin^2\theta}\right) W$$

and finally from (1),

$$F = \frac{l \sin\theta \cos\theta}{e - f \sin^2\theta} W$$

(2) A weightless rod AB of length l can turn freely about one end A in a vertical plane. A weight W is suspended from a point C of the rod; $AC = c$. A cord BD attached to the end B of the rod holds it in equilibrium in a horizontal position, the angle ABD being $\alpha = 150°$. Find the tension T of the cord and the resulting pressure A on the hinge at A.

(3) A cylinder of length $2l$ and radius r rests with the point A of the circumference of its lower base on a horizontal plane and with the point B of the circumference of its upper base against a vertical wall. The vertical plane through the axis of the cylinder contains the points A, B and is perpendicular to the intersection of the vertical wall with the horizontal plane. If there be no friction at A, B, what horizontal force F applied at A will keep the cylinder in equilibrium? When is this force $F = 0$?

(4) A weightless rod AB rests without friction on two planes inclined to the horizon at angles α, β, and carries a weight W at the point D. The intersection C of these planes is horizontal and normal to the vertical plane through AB. Find the inclination θ of AB to the horizon and the pressures at A and B.

(5) A weightless rod $AB = l$ can revolve in a vertical plane about a hinge at A; its other end B leans against a smooth vertical wall whose distance from A is $AD = a$. At the distance $AC = c$ from A a weight W is suspended. Find the horizontal thrust A_x at A and the normal pressures A_y and B at A and B.

(6) The same as (5) except that at B the rod rests on a smooth horizontal cylinder whose axis is at right angles to the vertical plane through AB. In which of the two problems is the horizontal thrust A_x at A least?

5. The general system of forces.

222. To reduce any system of forces acting on a rigid body to its most simple form the same methods are used as for complanar forces (comp. Art. 217)

Selecting as origin any point O rigidly connected with the body, let two equal and opposite forces $F, -F$ be applied at O, for every one of the given forces F. The effect of the given system of forces on the body is not changed by the introduction of these forces at O. But we may now regard the given force F acting at its point of application P as replaced by the equal and parallel force F at O, in combination with the couple formed by the original force F at P and the feroe $-F$ at O. All the forces of the given system are thus transferred to a common point of application O, and these forces at O can be replaced by a single resultant R, passing through O and represented in magnitude and direction by the geometric sum of the forces. In addition to this resultant R, we obtain as many couples $(F, -F)$ as there were forces given; and their resultant is found by geometrically adding the vectors of the couples (Art. 214).

Thus the given system of forces is seen to be equivalent to a resultant R in combination with a couple whose vector we shall call H; in other words, it has been proved that *any system of forces acting on a rigid body can be reduced to a single resultant force in combination with a single resultant couple.*

It follows at once that the geometrical **conditions of equilibrium** are

$$R = 0, \quad H = 0$$

223. Of the two geometrical elements representing a general system of forces, viz. the rotor R and the vector H, the for-

mer being merely the geometric sum of the forces, is independent of the point of reference O, while the vector H is in general different for different points of reference.

If the elements R, H for a point O are given, those for any other point O' can readily be found. It suffices to apply at O' equal and opposite forces R and $-R$. We then have R at O', and two couples, viz. the couple whose vector is H and the couple formed by R at O and $-R$ at O'; the resultant of the vectors of these two couples is the vector H' corresponding to O'. Here, as well as in the following articles, it is assumed that $R \neq 0$; when $R = 0$ the system reduces to a couple, the same whatever the point of reference.

If the new point of reference O' had been selected on the line l of the original resultant, no new couple would have been introduced, and H would not have been changed. But whenever the new point of reference O' is taken on a line l' different from l, the vector of the resultant couple H is changed.

By increasing the distance r between l and l' the moment Rr of the additional couple is increased. The effect of combining this additional couple Rr with H is, in general, to vary both the magnitude of the resulting vector H' and the angle ϕ it makes with the direction of the resultant R. It can be shown that the line l' of the new resultant can always be selected so as to reduce the angle ϕ to zero. The line l_0 for which $\phi = 0$, *i. e.* for which the vector H of the resultant couple is parallel to the resultant force R, is called the **central axis** of the given system of forces. We proceed to show how it can be found (comp. Art. 123).

224. Let the vector H be resolved at O into a component $H_0 = H \cos\phi$ along l, and a component $H_1 = H \sin\phi$, at

right angles to l (Fig. 57). In the plane passing through l at right angles to H_1, it is always possible to find a line l_0 parallel to l at a distance r_0 from l, such as to make $Rr_0 = -H_1$.

The line l_0 so determined is the central axis. For, if this line be taken as the line of the resultant R, the additional

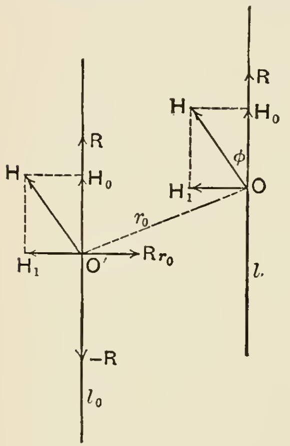

Fig. 57.

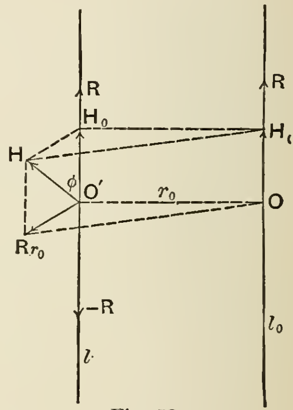

Fig. 58.

couple Rr_0 destroys the component H_1, so that the resulting couple H_0 has its vector parallel to R.

As the direction of the vector H is always changed in passing from line to line, there can be but one central axis for a given system of forces.

It appears from the construction of the central axis given above, that the vector of the resulting couple for this axis l_0 is $H_0 = H \cos\phi$; it is, therefore, less than for any other line.

It is instructive to observe how the vector H increases and changes its direction as we pass from the central axis l_0 to any parallel line l.

The transformation from l_0 to l requires the introduction of a couple whose vector Rr_0 (Fig. 58) is at right angles to the plane (l_0, l) and combines with H_0 to form the resulting couple H for l. As the distance r_0 of l from l_0 is increased, both the magnitude of H and the angle ϕ it makes with l increase, the angle ϕ approaching $\frac{1}{2}\pi$ as r_0 becomes infinite.

225. It is evident that since $H_0 = H \cos\phi$, the product $RH \cos\phi$ is a constant quantity for a given system of forces. It may be called an **invariant** of the system.

If the elements of reduction for the central axis R, H_0 be given, those for any parallel line l at the distance r_0 from the central axis are determined by the equations

$$H^2 = H_0^2 + R^2 r_0^2, \quad \tan\phi = \frac{Rr_0}{H_0}.$$

To sum up the results of the preceding articles, it has been shown that *any system of forces acting on a rigid body can be reduced, in an infinite number of ways, to a resultant R in combination with a couple H.* For all these reductions the magnitude, direction, and sense of the resultant R are the same, but the vector H of the couple changes according to the *position* assumed for the line of R. There is one, and only one, position of R, called the central axis of the system, for which the vector H is parallel to R and has at the same time its least value, H_0; this value H_0 is equal to the projection of any other vector H on the direction of the resultant R.

226. While, in general, a system of forces cannot be reduced to a single resultant, it can always be reduced to *two non-intersecting forces.* This easily follows by considering the system reduced to its resultant R and resulting couple H for any point O (Fig. 59). Let F, $-F$ be the forces, p the arm of the couple H, and place this couple so that one of the

forces, say $-F$, intersects R at O. Then, if R and $-F$ be replaced by their resultant F', the given system of forces is evidently equivalent to the two non-intersecting forces F, F' (compare Art. 216).

The two forces F, F' determine a tetrahedron $OABC$; and it can be shown that *the volume of this tetrahedron is constant and equal to one sixth of the invariant of the system* (Art. 225). The proof readily appears from Fig. 59. The volume of the

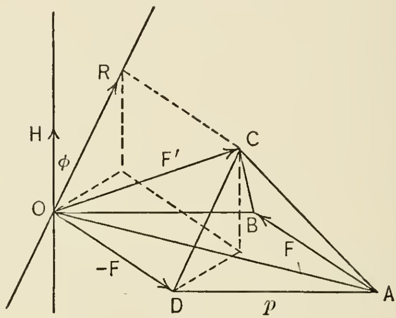

Fig. 59.

tetrahedron $OABC$ is evidently one half of the volume of the quadrangular pyramid whose vertex is C and whose base is the parallelogram $ODAB$. The area of this parallelogram is $Fp = H$; and the altitude of the pyramid is $= R\cos\phi$, being equal to the perpendicular let fall from the extremity of R on the plane of the couple; hence the volume of the tetrahedron

$$= \tfrac{1}{6}RH\cos\phi = \tfrac{1}{6}RH_0.$$

227. To *effect the reduction of a given system of forces analytically*, it is usually best to refer the forces F and their points of application P to a rectangular system of co-ordinates Ox, Oy, Oz (Fig. 60). Let x, y, z be the co-ordinates of P and X, Y, Z the components of F parallel to the axes.

To transfer these components to O and at the same time to introduce only couples whose vectors are parallel to the axes, we proceed in two steps. Thus to transfer, say X, we introduce at P', the foot of the perpendicular let fall from P on the plane zx, two equal and opposite forces X, $-X$; and we do the same thing at O. Then the single force X at P is replaced by the force X at O in combination with the two couples formed by X at P, $-X$ at P', and X at P', $-X$

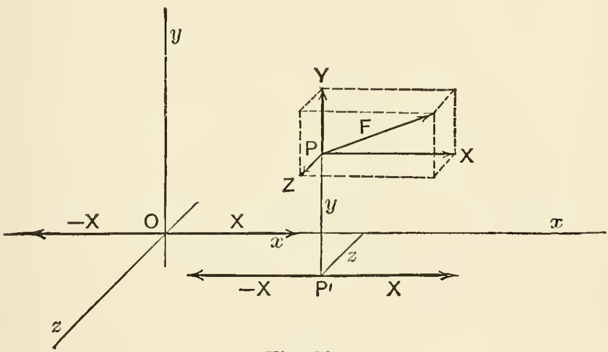

Fig. 60.

at O. The vector of the former couple is parallel to Oz, its moment is $-yX$; the negative sign being used because for a person looking on the plane of the couple from the positive side of the axis Oz the couple rotates clockwise. The vector of the latter couple is parallel to Oy, and its moment is zX.

The transfer of Y to the origin O requires the introduction of two couples, $-zY$ having its vector parallel to Ox and xY having its vector parallel to Oz.

Finally, transferring Z to O, we have to introduce the couples $-xZ$ with a vector parallel to Oy, and yZ with a vector parallel to Ox.

Thus each force F is replaced by three forces, X, Y, Z along the axes of co-ordinates and applied at O, in combination with three couples whose vectors are $yZ - zY$ parallel to Ox, $zX - xZ$ parallel to Oy, $xY - yX$ parallel to Oz.

228. If this be done for every force of the given system and the components having the same direction be added, the system will be found equivalent to the three rectangular forces

$$\Sigma X,\ \Sigma Y,\ \Sigma Z,$$

applied at O, together with the three couples

$$\Sigma(yZ - zY), \quad \Sigma(zX - xZ), \quad \Sigma(xY - yX),$$

whose vectors are at right angles.

The three forces can now be replaced by a single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

whose direction is determined by the angles α, β, γ which it makes with the axes Ox, Oy, Oz:

$$\cos\alpha = \frac{\Sigma X}{R}, \quad \cos\beta = \frac{\Sigma Y}{R}, \quad \cos\gamma = \frac{\Sigma Z}{R}.$$

In the same way the three couples can be replaced by a single resulting couple whose moment is

$$H = \sqrt{[\Sigma(yZ - zY)]^2 + [\Sigma(zX - xZ)]^2 + [\Sigma(xY - yX)]^2}.$$

229. Since R^2, as well as H^2, is thus found as the sum of three squares, each of these quantities can vanish only if the three squares composing it vanish separately. The **conditions of equilibrium of a rigid body** (Art. 222) are therefore expressed analytically by the following six equations:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma(yZ - zY) = 0, \quad \Sigma(zX - xZ) = 0, \quad \Sigma(xY - yX) = 0.$$

As the system of co-ordinates can be selected arbitrarily, the meaning of the first three equations is that the sum of the components of all the forces along any three lines not parallel to the same plane must vanish. The last three equations express that the sum of the moments of all the forces about any three axes not parallel to the same plane must also vanish.

The *moment of a force about an axis* must be understood as meaning the moment of its projection on a plane at right angles to the axis with respect to the point of intersection of the axis with the plane. This definition is in accordance with the somewhat vague notion of the moment of a force as representing its "turning effect." For, if we regard the force as acting on a rigid body with a fixed axis, the force can be resolved into two components, one parallel, the other perpendicular, to the axis; the former component evidently does not contribute to the turning effect which is, therefore, measured by the moment of the latter alone.

230. The *equations of the central axis* (Art. 223) can be found by a transformation of co-ordinates.

Let the system be reduced for any point O to its resultant R, whose rectangular components we denote by

$$A = \Sigma X, \quad B = \Sigma Y, \quad C = \Sigma Z,$$

and to the vector H of its resulting couple with the components

$$L = \Sigma(yZ - zY), \quad M = \Sigma(zX - xZ), \quad N = \Sigma(xY - yX).$$

If a point O' whose co-ordinates are ξ, η, ζ be taken as new point of reference and the co-ordinates of any point with respect to parallel axes through O' be denoted by x', y', z', we have $x = \xi + x'$, $y = \eta + y'$, $z = \zeta + z'$. Substituting these values, we find

$$L = \Sigma[(\eta + y')Z - (\zeta + z')Y] = \eta\Sigma Z - \zeta\Sigma Y + \Sigma(y'Z - z'Y)$$
$$= \eta C - \zeta B + L',$$

where L' is the x-component of the couple H' resulting for O' as point of reference. Similar expressions hold for M and N. The components of H' are therefore

$$L' = L - \eta C + \zeta B,\ M' = M - \zeta A + \xi C,\ N' = N - \xi B + \eta A;$$

and its direction cosines are

$$\lambda = \frac{L'}{H'},\quad \mu = \frac{M'}{H'},\quad \nu = \frac{N'}{H'}.$$

The central axis being defined (Art. 223) as that line for which the vector of the resulting couple is parallel to the direction of the resultant, the point $O'(\xi, \eta, \zeta)$ will lie on the central axis if the direction cosines of H' are equal to those of R, viz. to $\alpha = A/R$, $\beta = B/R$, $\gamma = C/R$. Hence the equations of the central axis are

$$\frac{L'}{A} = \frac{M'}{B} = \frac{N'}{C},$$

that is,

$$\frac{L - \eta C + \zeta B}{A} = \frac{M - \zeta A + \xi C}{B} = \frac{N - \xi B + \eta A}{C}.$$

5. Constraints; friction.

231. It has been shown in Art. 229 that the number of the conditions of equilibrium is six, for a rigid body that is perfectly free. This number will be diminished whenever the body is subject to conditions restricting its possible motions. Such conditions, or **constraints,** may be of various kinds; the body may have a fixed point, or a fixed axis, or one of its points may be constrained to move along a given curve or to remain on a given surface, etc.

Now *a free point is said to have three degrees of freedom* because its position is determined by three co-ordinates. One conditional equation between its co-ordinates restricts the point to the surface represented by that equation; it has then one constraint and two degrees of freedom. Two conditions restrict the point to a curve, viz. the intersection of the two surfaces represented by the two equations of condition; the point then has two constraints and one degree of freedom.

The position of a rigid body is determined by the position of any three of its points, not in a line, *i. e.* by nine co-ordinates between which, however, there exist three conditions, expressing the constancy of the distances of the three points. *A free rigid body has therefore six degrees of freedom*, since six independent quantities determine its position.

The most general instantaneous state of motion that a free rigid body can have is a twist, or screw-motion (Art 123), consisting of an angular velocity about a certain axis and a linear velocity along this axis; each of these velocities has three components along the rectangular axes, and these six components can be regarded as the six independent possible motions of the body, on account of which it is said to have six degrees of freedom.

Equilibrium will exist only when these six possible motions are prevented; hence there must be six conditions of equilibrium.

232. We proceed to consider some forms of constraint and the corresponding changes in the equations of equilibrium.

It is often convenient in dynamics to replace such restraining conditions by forces, usually called **reactions.** Whenever it is possible to introduce such forces having the

same effect as the given conditions, the body may be regarded as free, and the general equations of equilibrium can be applied.

233. Rigid Body with a Fixed Point. A body that is free to turn about a fixed point A can be regarded as free if the reaction A of this point be introduced and combined with the other forces acting on the body.

Let A_x, A_y, A_z be the components of A; then, taking the fixed point A as origin, the six equations of equilibrium (Art. 229) are

$$\Sigma X + A_x = 0, \qquad \Sigma Y + A_y = 0, \qquad \Sigma Z + A_z = 0,$$
$$\Sigma(yZ - zY) = 0, \quad \Sigma(zX - xZ) = 0, \quad \Sigma(xY - yX) = 0.$$

The first three of these equations serve to determine the reaction of the fixed point; the last three are the actual conditions of equilibrium corresponding to the three degrees of freedom of a body with a fixed point.

Hence, *a rigid body having a fixed point is in equilibrium if the sum of the moments of all the forces vanishes for any three non-complanar axes passing through the fixed point.*

234. Rigid Body with a Fixed Axis. A body with a fixed axis has but one degree of freedom; indeed, the only possible motion consists in rotation about this axis.

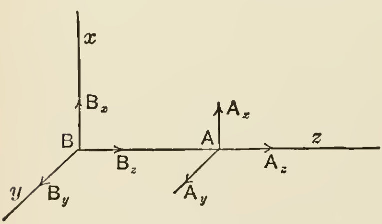

Fig. 61.

An axis is fixed as soon as two of its points, say A, B, are fixed. Hence, after introducing the reactions A_x, A_y, A_z, B_x, B_y, B_z, of these points, the body can be regarded as free. If the point B be taken as origin, the line BA as axis of z (Fig. 61), the equations of equilibrium become

$$\Sigma X + A_x + B_x = 0, \qquad \Sigma Y + A_y + B_y = 0, \qquad \Sigma Z + A_z + B_z = 0,$$
$$\Sigma(yZ - zY) - A_y a = 0, \qquad \Sigma(zX - xZ) + A_x a = 0,$$
$$\Sigma(xY - yX) = 0,$$

where $a = BA$.

The last of the six equations is the only independent condition of

equilibrium of the constrained body; the first five determine A_x, B_x, A_y, B_y, $A_z + B_z$. The two z-components cannot be found separately, since they act in the same straight line.

Hence, *a rigid body having a fixed axis is in equilibrium if the sum of the moments of all the forces vanishes for the fixed axis.*

235. If, in the preceding article, the axis be not absolutely fixed, but only fixed in direction so that *the body can rotate about the axis and also slide along it,* we have evidently

$$A_z = 0, \quad B_z = 0;$$

hence, by the third equation of equilibrium.

$$\Sigma Z = 0,$$

as an additional condition of equilibrium.

The body has in this case two degrees of freedom.

236. Rigid Body with a Fixed Plane. A body constrained to slide on a fixed plane (that is, to move so that the paths of all its points lie in parallel planes) has three degrees of freedom. At every point of contact between the body and the plane, the latter exerts a reaction. As all these reactions are parallel, they are equivalent to a single resultant N. Taking the fixed plane as the plane xy, N will be parallel to the axis of z; hence, if a, b, 0 be the co-ordinates of its point of application, the six equations of equilibrium are

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z + N = 0,$$
$$\Sigma(yZ - zY) + bN = 0, \qquad \Sigma(zX - xZ) - aN = 0,$$
$$\Sigma(xY - yX) = 0.$$

The third, fourth, and fifth equations determine the reaction N and the co-ordinates a, b of its point of application. The three other equations are the actual conditions of equilibrium; they agree, of course, with the three conditions of equilibrium of a plane system as found in Art. 219.

If there be not more than three points of contact (or supports) between the body and the fixed plane, the reactions of these points can be found separately. Let A_1, A_2, A_3 be the three points of contact; N_1, N_2, N_3 the required reactions; a_1, b_1, a_2, b_2, a_3, b_3 the co-ordinates of A_1, A_2, A_3; then N must be resolved into three parallel forces passing through these points, and the conditions are

$$N_1 + N_2 + N_3 = N,$$
$$a_1N_1 + a_2N_2 + a_3N_3 = aN,$$
$$b_1N_1 + b_2N_2 + b_3N_3 = bN.$$

These three equations always determine N_1, N_2, N_3. For if the determinant of the coefficients of N_1, N_2, N_3 vanished,

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0,$$

the three points A_1, A_2, A_3 would lie in a straight line, and hence the body would not be properly constrained.

The reactions become indeterminate whenever there are more than three points of contact.

237. Friction. The reaction between two surfaces in contact has so far been regarded as directed along the common normal of the surfaces (Art. 195). If this is true the surfaces are said to be *perfectly smooth.*

The surfaces of physical bodies are *rough*, *i. e.* they present small elevations and depressions; when two such surfaces are "in contact" the projections of one will more or less enter into depressions of the other; the greater the normal pressure between the surfaces, the more will this be the case. Hence when a tangential force acting on one of the bodies tends to *slide* its surface over that of the other body, a resistance will be developed whose magnitude must depend on the roughness of the surfaces and on the normal pressure between them. This resistance is called the **force of sliding friction,** or simply the **friction.**

The study of friction belongs properly to applied mechanics, and will here only be touched upon very briefly.

238. Imagine a body resting with a plane surface on a horizontal plane. Let a small horizontal force P be applied at its centroid (which is supposed to be situated so low that

the body is not overturned), and let the force P be gradually increased until motion ensues. At any instant before motion sets in, the friction is equal to the value of P at that instant. The value of P at the moment when motion just begins is equal and opposite to the *frictional resistance* F between the surfaces at this moment, and this resistance is called the **limiting static friction.**

Careful experiments with dry solids in contact have shown this force to be subject to the following laws:

(1) *The magnitude of the limiting friction F bears a constant ratio to the normal pressure N between the surfaces in contact*; that is

$$F = \mu N,$$

where μ is a constant depending on the condition and nature of the surfaces in contact. This constant which must be determined experimentally for different substances and surface conditions is called the **coefficient of static friction.** It is in general a proper fraction; for perfectly smooth surfaces $\mu = 0$.

(2) *For a given normal pressure the limiting static friction, and hence the coefficient of static friction, is independent of the area of contact, provided the pressure be not so great as to produce cutting or crushing.*

239. The frictional resistance between two surfaces in relative motion is called **kinetic friction.** It is subject, in addition to the two laws just mentioned, to the third law:

(3) *For moderate velocities, kinetic friction is nearly independent of the velocities of the bodies in contact.*

The coefficient of static friction is somewhat greater than that of kinetic friction. A slight jarring will often reduce the coefficient from its static to its kinetic value.

It must not be forgotten that these so-called **laws of friction** are experimental laws, and therefore true only approximately and within the limits of the experiments from which they were deduced. When the relative velocity of the surfaces in contact is high, or when, as is usually the case in machinery, a lubricating material is introduced between the two surfaces, the frictional resistance is found to depend on a number of other circumstances, such as the temperature, the form of the surfaces, the velocity, the nature of the lubricator, etc. Indeed, when the supply of the lubricant is sufficient, the two solid surfaces are kept by it out of actual contact; the coefficient of friction in this case varies with the pressure, area of contact, velocity, and temperature.

240. Consider again a body resting on a horizontal plane (Fig. 62) and acted upon by a horizontal force P just large enough to equal the limiting friction F. The normal reaction N of the plane is equal and opposite to the weight W. The body is thus in equilibrium under the action of the two pairs of equal and opposite forces; but motion will ensue as soon as P is increased. If P be decreased, F will decrease at the same rate, so that the equilibrium remains undisturbed.

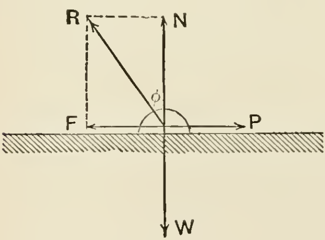

Fig. 62.

The force of friction F can be combined with the normal reaction N to form a resultant,

$$R = \sqrt{F^2 + N^2} = \sqrt{P^2 + W^2},$$

which represents the *total reaction* of the horizontal plane.

If ϕ be the angle between N and R when F has its limiting value $F = \mu N$ (Art. 238), we have, since $\tan\phi = F/N$,

$$\tan\phi = \mu.$$

The angle ϕ thus furnishes a graphical representation for the coefficient of friction μ; it is called the **angle of friction.**

241. If the plane be not horizontal, but inclined to the horizon at an angle θ, the weight W of the body (regarded as a particle) resting on the plane can be resolved into a component $W\sin\theta$ along the plane, and a component $W\cos\theta$ perpendicular to it (Fig 63). Hence, if no other forces act on the body it will be in equilibrium, provided the component $W\sin\theta$ be not greater than the limiting friction $F = \mu W\cos\theta$. The limiting condition of equilibrium is therefore,

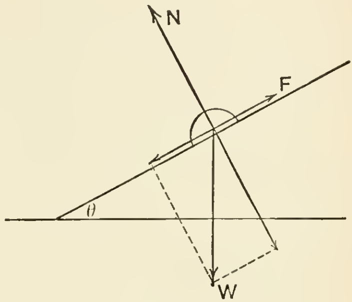

Fig. 63.

$$\mu W\cos\theta = W\sin\theta, \quad \text{or} \quad \mu = \tan\theta;$$

in other words, if the angle θ be gradually increased, the body will not slide down the plane until $\theta > \phi$. This furnishes an experimental method of determining the angle of friction ϕ, which on this account is sometimes called the **angle of repose.**

242. A particle P (Fig. 64) will be in equilibrium on any rough surface, if the total reaction of the surface, *i. e.* the resultant R of the normal reaction N and the friction F, is equal and opposite to the resultant R' of all the other forces acting on the particle.

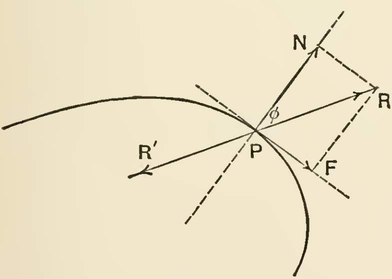

Fig. 64.

The limiting value of the angle between N and R is ϕ so that the particle can be in equilibrium only if the resultant R' makes with the normal an angle $\overline{\overline{<}}\phi$. Hence, if about the normal PN as axis, and with P as vertex, a cone be described whose vertical angle is 2ϕ, the condition of equilibrium is that R' must lie within this cone. The cone is called the **cone of friction.**

243. Exercises.

(1) A weight W is to be hauled along a horizontal plane, the coefficient of friction being $\mu = \tan\phi$. Determine the required tractive force P if it is to act at an inclination α to the horizon, and show that this force is least when $\alpha = \phi$.

(2) A particle of weight W is in equilibrium on a rough plane inclined to the horizon at an angle θ, under the action of a force P parallel to the plane along its greatest slope. Determine P: (a) when $\theta > \phi$, (b) when $\theta = \phi$, (c) when $\theta < \phi$, $\phi = \tan^{-1}\mu$ being the angle of friction.

(3) Solve Ex. (2) (a) graphically by means of the friction angle and determine what part of P is required to overcome friction.

(4) A body weighing 240 pounds is pulled up a plane inclined at 45°, by means of a rope. If $\mu = 1/3$, find the tension of the rope. What portion of it is due to friction?

(5) A homogeneous straight rod $AB = 2l$ of weight W rests with one end A on a horizontal floor, with the other end B against a vertical wall whose plane is at right angles to the vertical plane of the rod. If there be friction of angle ϕ at both ends, determine the limiting position of equilibrium.

(6) A straight homogeneous rod $AB = 2l$, of weight W, rests with the lower end A on a rough horizontal plane and with the point C ($AC = c$) on a smooth cylindrical support. The rod is in equilibrium when inclined at a given angle θ to the horizon; determine the coefficient of friction at A and the reactions at A and C.

(7) If in Ex. (6) there be friction both at A and C, the friction angle ϕ being the same, find the position of equilibrium and the reactions at A and C.

(8) A solid homogeneous hemisphere is placed with its curved surface on a rough inclined plane; investigate the conditions of equilibrium.

CHAPTER XII.

THEORY OF ATTRACTIVE FORCES.

1. Attraction.

244. Among the various kinds of forces introduced in physics for describing and interpreting natural phenomena, forces of attraction and repulsion occupy a most prominent place.

According to **Newton's law** (the law of universal or cosmical gravitation, the "law of nature") every particle of matter attracts every other such particle with a force proportional to the masses and inversely proportional to the square of the distance of the particles, and this force acts along the line joining the particles.

Thus, if m, m' are the masses of the particles, r their distance, and κ a constant, the force with which m attracts m' and m' attracts m is

$$F = \kappa \frac{mm'}{r^2}.$$

Each particle is here regarded as a mass concentrated at a point; otherwise we could not speak of the distance of the particles and of the line joining them (comp. Art. 156). As the distance r approaches zero, the magnitude of the force F becomes infinite and its direction indeterminate.

245. In the theory of gravitation, the masses m, m' are essentially positive. The constant κ, called the **constant of gravitation,** evidently represents the force with which two particles, each of mass 1, attract each other when at the

distance 1. It is a physical constant to be determined by experiment, and its numerical value depends on the units of measurement adopted for mass, length, and time.

What can be directly observed is of course not the force itself, but the acceleration it produces. Dividing the force F (Art. 244) by the mass m of the attracted particle on which it acts we have the *acceleration* j produced by the force with which m' attracts m at the distance r from m:

$$j = \kappa \frac{m'}{r^2}.$$

246. It will be shown later (Art. 253) that the attraction of a homogeneous sphere at any external point is the same as if the mass of the sphere were concentrated at its center. Hence if m' be the mass of the earth (here regarded as a homogeneous sphere) the acceleration it produces in any mass m concentrated at a point P above its surface, at the distance $OP = r$ from the center O, is $j = \kappa m'/r^2$. Now for points near the earth's surface this acceleration is known from experiments; it is the acceleration g of a body falling in vacuo (apart from the slight effect due to the earth's rotation, see Arts. 334, 461). Hence, taking the radius of the earth as $r = 6.37 \times 10^8$ cm., its mean density as $\rho = 5\frac{1}{2}$, and $g = 980$ cm./sec.^{2}, we find in C.G.S. units

$$\kappa = 6.7 \times 10^{-8}.$$

This, then, is the force in dynes with which two masses, of 1 gram each, would attract each other if concentrated at two points 1 cm. apart.

Conversely, the mean density of the earth can be found with considerable accuracy by a direct experimental determination of the attraction of gravitation between two given masses at a given distance.

247. Exercises.

(1) With $r = 3960$ miles, $g = 32$ ft./sec.^{2}, $\rho = 5\frac{1}{2}$, show that the attraction between two masses of 1 lb. each, at a distance of 1 ft., is equal to the weight of 0.33×10^{-10} lb.

(2) In astronomy and in the general theory of attraction it is convenient to take the unit of mass so that $\kappa = 1$. Show that this *astronomical unit of mass*, *i. e.* the mass which acting on an equal mass at unit distance would produce unit acceleration, is $= 1/\kappa$.

(3) Show that $\kappa = 1$ if, with the ordinary unit of mass, the unit of time be taken as 3862 sec. This has been called the "natural hour."

248. If more than two particles are given the forces of attraction exerted on any one of the particles, m, being concurrent are equivalent to a single resultant. This resultant, divided by the mass m of the attracted particle, is called the **attraction at the point** P where m is situated.

If, instead of a finite number of particles, any continuously distributed masses of one, two, or three dimensions (Art. 155) are given they can be resolved into elements which in the limit can be regarded as particles. The first problem in the theory of attraction consists in *determining the attraction at any point, due to any given masses.*

Notice that the "attraction at any point," as thus defined, has the dimensions of an acceleration and not of a force.

Let $P(x, y, z)$ be the attracted point of mass 1, dm' an element of the attracting masses at $Q(x', y', z')$, $PQ = r$ the distance of these points; then the attraction at P due to dm' is $\kappa dm'/r^2$, and if α, β, γ are its direction cosines, its components are $\kappa\alpha dm'/r^2$, $\kappa\beta dm'/r^2$, $\kappa\gamma dm'/r^2$. Hence the attraction A at P, due to all the given masses, has the rectangular components:

$$X = \kappa\int\frac{\alpha dm'}{r^2}, \quad Y = \kappa\int\frac{\beta dm'}{r^2}, \quad Z = \kappa\int\frac{\gamma dm'}{r^2},$$

with $r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$, the integrations extending over all the masses. The attraction A itself and its direction cosines l, m, n are:

$$A = \sqrt{X^2 + Y^2 + Z^2}, \quad l = \frac{X}{A}, \quad m = \frac{Y}{A}, \quad n = \frac{Z}{A}.$$

It is in general most convenient to take the attracted point P as origin so that $r^2 = x'^2 + y'^2 + z'^2$

249. If the point P were situated within the attracting masses, $1/r^2$ would become infinite within the limits of integration; hence a special investigation would be necessary to determine whether the integrals representing X, Y, Z have a meaning. It can be shown without difficulty in the case of three-dimensional masses that the integrals have a meaning and represent the attraction even at an internal point P. But for the sake of simplicity, we here confine ourselves to the *external field.* In other words, we assume, when nothing is said to the contrary, that P is an *external point, i. e.* a point such that a sphere can be described about it such as not to contain within it any portion of the attracting matter (except the unit mass at P itself).

250. The problem of attraction can be generalized in various ways. Thus, in electricity and magnetism, we have to consider both positive and negative masses, and the force may be a repulsion as well as an attraction. The force between two electric charges as well as that between two magnetic poles follows Newton's law (Art. 244); *i. e.* the force is directly proportional to the charges, or pole-strengths, and inversely proportional to the square of the distance. But the constant κ has a very different value. It is customary to select the units of electric charge and magnetic pole-strength so that $\kappa = 1$.

It is sometimes necessary to consider forces that do not follow Newton's law of the distance. Indeed, Newtonian attraction is merely a particular case of the more general type of force

$$F = \kappa m m' f(r),$$

viz. the case when $f(r) = 1/r^2$.

251. Spherical shell. *The attraction due to a mass spread uniformly over a sphere is zero at any point within the sphere, while at any outside point it is the same as if the mass were concentrated at the center.*

Geometrical method. (*a*) *Attraction at an inside point.* Let C be the center, a the radius of the sphere (Fig. 65). A cone of vertex P and *solid angle* $d\omega$ (*i. e.* cutting out an area element $d\omega$ on the sphere of radius 1 about P as center) cuts out on the given sphere a surface element $d\sigma$ at Q and a surface element $d\sigma'$ at Q'. It will be shown that the mass elements on these surface elements produce equal and opposite attractions at P. As the whole sphere can thus be divided into pairs of elements whose attractions at P balance it follows that the attraction at P is zero.

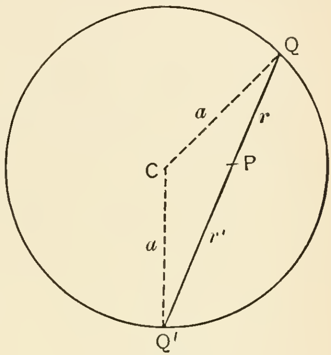

Fig. 65.

Put $PQ = r$, $PQ' = r'$; on the sphere of radius r about P the cone cuts out an element $r^2 d\omega$ at Q, and we have evidently $d\sigma = r^2 d\omega / \cos CQP$; hence if the surface density is ρ', the mass on $d\sigma$ is $\rho' r^2 d\omega / \cos CQP$, and the attraction at P due to this mass is $\kappa\rho' d\omega / \cos CQP$. In the same way we find that the mass on $d\sigma'$ at Q' produces at P the attraction

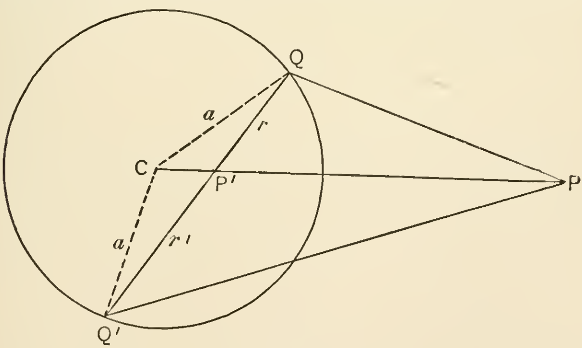

Fig. 66.

$\kappa\rho' d\omega / \cos CQ'P$. As for the sphere the angles CQP and $CQ'P$ are equal, the attractions are equal.

(*b*) *Attraction at an outside point.* Let P' (Fig. 66) be the point

inverse to P with respect to the given sphere, *i. e.* the point P' on CP such that if $CP = p$, $CP' = p'$, we have

$$pp' = a^2.$$

The extremities Q, Q' of any chord through P' determine with C, P, P' two pairs of similar triangles: CQP' and CPQ, $CQ'P'$ and CPQ'; for, each pair has the angle at C in common and the sides including the equal angles proportional owing to the relations $pp' = a^2$, $CQ = CQ' = a$. It follows that $\measuredangle CQP' = CPQ$, $\measuredangle CQ'P' = CPQ'$; hence, as the triangle QCQ' is isosceles, the line CP bisects the angle QPQ'.

With the aid of these geometrical properties it can be shown that equal attractions are produced at P by the masses on the elements $d\sigma$ at Q and $d\sigma'$ at Q', cut out by a cone of solid angle $d\omega$ with vertex at the point P' inverse to P. For the mass elements at Q, Q' we have as in the case (a):

$$dm = \rho' d\sigma = \rho' \frac{r^2 d\omega}{\cos CQP'}, \quad dm' = \rho' d\sigma' = \rho' \frac{r'^2 d\omega}{\cos CQ'P'},$$

where $r = P'Q$, $r' = P'Q'$. Hence the corresponding attractions at P are:

$$\frac{\kappa\rho' r^2 d\omega}{PQ^2 \cos CQP'}, \quad \frac{\kappa\rho' r'^2 d\omega}{PQ'^2 \cos CQ'P'},$$

and these are equal, since for the sphere $\measuredangle CQP' = CQ'P'$, and the similar triangles give

$$\frac{r}{PQ} = \frac{a}{p}, \quad \frac{r'}{PQ'} = \frac{a}{p}.$$

As shown above, these attractions make equal angles with PC; hence their components along this line are equal while their components at right angles to CP are equal and opposite. The two elements $d\sigma$ at Q and $d\sigma'$ at Q' produce therefore together at P an attraction along PC equal to

$$\frac{2\kappa\rho' a^2 d\omega}{p^2}.$$

The coefficient of $d\omega$ is constant; the summation over the unit sphere gives $\int d\omega = 2\pi$, since a double cone was used. Hence the total attraction at P is

$$A = 4\pi\kappa\rho' \frac{a^2}{p^2} = \kappa \frac{m'}{p^2},$$

where $m' = 4\pi\rho' a^2$ is the whole mass on the sphere. This shows that the attraction is the same as if this mass were concentrated at the center of the sphere.

(c) *Attraction at a point on the sphere.* If the point P approaches the surface from within the attraction remains constantly zero; if P approaches the surface from without the attraction approaches the limit $\kappa m'/a^2 = 4\pi\kappa\rho'$. At a point P on the sphere (Fig. 67) the attraction can be shown to be

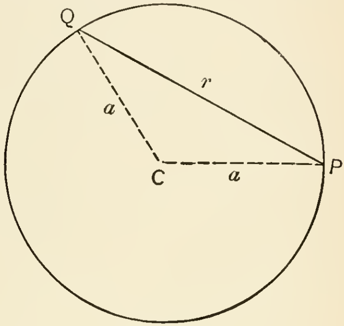

Fig. 67

$$A = 2\pi\kappa\rho'.$$

For, the mass on $d\sigma$ at Q is $\rho' d\sigma = \rho' r^2 d\omega / \cos CQP$; its attraction at P is $= \kappa\rho' d\omega/\cos CQP$, and as the angles at P and Q are equal, the projection of this attraction on PC is $\kappa\rho' d\omega$. As P lies on the surface, $\int d\omega = 2\pi$; hence the total attraction is $= 2\pi\kappa\rho'$.

The attraction exerted by the whole mass on the mass element $\rho' d\sigma$ situated at P is of course $= 2\pi\kappa\rho'^2 d\sigma$.

252. *Analytical method.* Whether P lies inside or outside the sphere we take P as origin, PC as polar axis, and put $PQ = r$, $\measuredangle PCQ = \phi$,

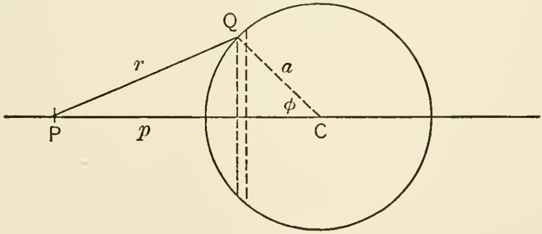

Fig. 68.

Q being any point of the sphere (Fig. 68), and as before $CP = p$, $CQ = a$. As mass element take the mass $\rho' \cdot 2\pi a \sin\phi \cdot a d\phi$, contained between the plane through Q at right angles to PC and an infinitely near parallel plane. The attraction produced at P by this element is directed along PC and

$$= 2\pi\kappa\rho' a^2 \sin\phi d\phi \cdot \frac{\cos CPQ}{r^2} = 2\pi\kappa\rho' a^2 \sin\phi d\phi \cdot \frac{p - a\cos\phi}{r^3},$$

where

$$r^2 = a^2 + p^2 - 2ap\cos\phi,$$

and hence

$$rdr = ap\sin\phi\, d\phi.$$

Substituting for $a\sin\phi\, d\phi$ and $a\cos\phi$ their expressions from these relations we find for the attraction of the ring element at P:

$$\pi\kappa\rho' \frac{a}{p^2} \frac{p^2 - a^2 + r^2}{r^2}\, dr.$$

(*a*) *For an inside point* P we have $p < a$, and the limits of integration for r are from $a - p$ to $a + p$. Hence the resultant attraction at P is

$$A = \pi\kappa\rho' \frac{a}{p^2} \int_{a-p}^{a+p} \left(\frac{p^2 - a^2}{r^2} + 1\right) dr = \pi\kappa\rho' \frac{a}{p^2} \left(\frac{a^2 - p^2}{r} + r\right)_{a-p}^{a+p} = 0.$$

For an outside point P we have $p > a$ and the limits are from $p - a$ to $p + a$; hence

$$A = \pi\kappa\rho' \frac{a}{p^2} \left(\frac{a^2 - p^2}{r} + r\right)_{p-a}^{p+a} = 4\pi\kappa\rho' \frac{a^2}{p^2} = \kappa \frac{m'}{p^2}.$$

253. From the results of Arts. 251, 252, it readily follows that the attraction due to a homogeneous **solid shell** (mass between two concentric spheres) is zero within the hollow of the shell, while at an outside point it is the same as if the mass were concentrated at the center. It suffices to resolve the shell into concentric shells of infinitesimal thickness da and put $\rho' da = \rho$, the volume density.

In particular, for a *homogeneous solid sphere* of radius a and volume density ρ the attraction at the distance $p > a$ from the center is

$$A = \kappa \frac{m'}{p^2} = \frac{4}{3} \pi\kappa\rho' \frac{a^3}{p^2}.$$

254. Exercises.

(1) Show that the results of Art. 253 hold for a solid shell whose density is any function of the distance from the center.

(2) By Art. 252, the attraction due to a mass distributed uniformly over a sphere when considered as a function of p has a point of discontinuity; illustrate this by a sketch.

(3) Prove that the attraction at the center due to a mass distributed uniformly along a circular arc of angle 2α and radius a is $= 2\kappa\rho'' \sin\alpha/a$; show that a mass equal to that of the chord, if it had the same density

ρ'', placed at the midpoint of the arc, would produce the same attraction at the center.

(4) Prove that the attraction of a homogeneous rectilinear segment A_1A_2, at a point P whose perpendicular distance PO from A_1A_2 makes the angles θ_1, θ_2 with PA_1, PA_2, bisects the angle A_1PA_2 and has the value $2\kappa\rho'' \sin\frac{1}{2}(\theta_2 - \theta_1)/p$. Show that the arc of the circle of radius $PO = p$ about P, bounded by PA_1 and PA_2, if of the same density, produces at P the same attraction.

(5) Show that in any plane through A_1A_2 the confocal hyperbolas having A_1A_2 as foci are the lines of force in the *field* of the rectilinear segment; *i. e.* they have the property that the attraction at any point P is tangent to the hyperbola through P.

(6) Show that for a homogeneous rod of infinite length the attraction at any point is normal to the rod and inversely proportional to the distance from the rod. Hence show that the attraction due to a homogeneous circular cylinder, of radius a and infinite length, at any point P at the distance $PC = p > a$ from the axis, is $= 2\pi\kappa\rho a^2/p$.

(7) Prove that the attraction due to a mass spread uniformly over the area of a circle of radius a, at a point P on the axis of the circle, at the distance $PC = p$ from the center C, is $= 2\pi\kappa\rho'(1 - p/\sqrt{a^2 + p^2})$.

(8) Two parallel homogeneous straight rods of equal density ρ'' are placed so that the line joining their midpoints is at right angles to each; if their lengths are $2a$, $2b$, and their distance is c, find their *mutual attraction*, *i. e.* the force required to hold them apart.

(9) Show that the attraction exerted by a homogeneous right circular cone of vertical angle 2α and height h, at the vertex, is $= 2\pi\kappa\rho h(1 - \cos\alpha)$. Show that the same expression holds for a frustum of height h and angle 2α.

(10) Two equal circular disks, of radius a, are placed at right angles to the line joining their centers whose distance is c. If one attracts while the other repels, determine the resultant force at a point P on the line of the centers, at a distance p from the nearer center. What becomes of this force when c is indefinitely diminished?

(11) Show that the attraction of a homogeneous solid hemisphere at a point on its edge is $= \frac{2}{3}\kappa\rho a\sqrt{\pi^2 + 4}$, and that it is inclined to the base at an angle of about $32\frac{1}{2}°$.

2. The potential.

255. As shown in Art. 248, the determination of the attraction, due to given masses, at any particular point P is a mere problem of integration. The next problem that presents itself in the theory of attraction is to express the attraction A as a function of the point P, or rather the components X, Y, Z of A as functions of the co-ordinates x, y, z of P, and to study the nature of these functions. The solution of this problem is greatly facilitated by observing that there exists a function U, known as the **potential** of the given masses, which has the property that *the components of A are its first partial derivatives*:

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}.$$

A function having this property may exist for forces that are not Newtonian attractions; it is then called a *force-*

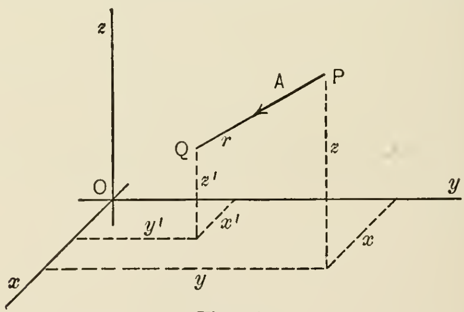

Fig. 69.

function. Forces for which a force-function exists are called *conservative forces.*

256. Let us consider the most simple case of Newtonian attraction, viz. the *field generated by a single particle m'*, situated at Q (Fig. 69). The attraction at $P(x, y, z)$, due

to m' at $Q(x', y', z')$ is $A = \kappa m'/r^2$, where $r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$. As this attraction has the sense from P toward Q, its direction cosines are $-(x - x')/r$, $-(y - y')/r$, $-(z - z')/r$; hence the rectangular components of the attraction are:

$$X = -\kappa m' \frac{x - x'}{r^3}, \quad Y = -\kappa m' \frac{y - y'}{r^3}, \quad Z = -\kappa m' \frac{z - z'}{r^3}.$$

It is easily verified that these expressions are the partial derivatives with respect to x, y, z of one and the same function, viz.

$$U = \frac{\kappa m'}{r};$$

this then is the *potential of a single particle* m'.

257. Notice that this function is one-valued and continuous throughout the whole of space, except at Q where it has a simple pole (*i. e.* U becomes infinite like $1/r$ for $r = 0$), and that it vanishes at infinity. The same properties hold for all derivatives of U except that Q becomes a pole of higher order.

For the projection of the attraction A on any direction s we have

$$A_s = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds} + \frac{\partial U}{\partial z}\frac{dz}{ds} = \frac{dU}{ds};$$

i. e. the s-component of A is the s-derivative of U.

For the second x-derivative of U we have since $\partial r/\partial x = (x - x')/r$:

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2} = \frac{\partial X}{\partial x} &= -\kappa m'\left(\frac{1}{r^3} - 3\frac{x - x'}{r^4}\frac{\partial r}{\partial x}\right)\\ &= -\kappa m'\left[\frac{1}{r^3} - \frac{3(x - x')^2}{r^5}\right];\end{aligned}$$

and similarly:

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial Y}{\partial y} = -\kappa m'\left[\frac{1}{r^3} - \frac{3(y-y')^2}{r^5}\right],$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial Z}{\partial z} = -\kappa m'\left[\frac{1}{r^3} - \frac{3(z-z')^2}{r^5}\right].$$

Adding and observing that $(x-x')^2 + (y-y')^2 + (z-z')^2 = r^2$ we find

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

This equation, satisfied by the potential at every point excepting the point Q where the attracting mass m' is situated, is known as **Laplace's equation,** or the **potential equation.**

258. These results are readily generalized. If the field is due to any finite number of particles m_1', m_2', $\cdots$ at the distances r_1, r_2, $\cdots$ from the attracted point P, their potential is defined as

$$U = \frac{\kappa m_1'}{r_1} + \frac{\kappa m_2'}{r_2} + \cdots = \Sigma \frac{\kappa m'}{r}.$$

If the field is due to continuous masses their potential is

$$U = \kappa \int \frac{dm'}{r}.$$

For, as the limits of integration are constant the derivatives of U with respect to x, y, z can be found by differentiating under the integral sign; we have therefore, at any rate at any external point P:

$$\frac{\partial U}{\partial x} = -\kappa \int \frac{x-x'}{r^3}\,dm', \quad \frac{\partial U}{\partial y} = -\kappa \int \frac{y-y'}{r^3}\,dm',$$

$$\frac{\partial U}{\partial z} = -\kappa \int \frac{z-z'}{r^3}\,dm',$$

where the right-hand members are evidently the components X, Y, Z of the attraction at P.

For masses of finite density and not extending to infinity it is not difficult to show that the function U has a single definite finite value at every point P external (and even internal) to the given masses and that it is a continuous function of x, y, z.

As in Art. 257 it can be shown that, at any external point, U satisfies Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

259. The potential is a *scalar* point-function; *i. e.* it is not a vector, but its value at any point is given by a single real number.

The locus of those points at which the potential U has a constant value c, *i. e.* the surface

$$U = c,$$

is called an **equipotential surface** (level, or equilibrium, surface).

As the first derivatives of U with respect to x, y, z are on the one hand equal to the components of the attraction while, on the other, they are proportional to the direction cosines of the normal to the surface $U = c$, it follows that *the attraction A at any external point P is normal to the equipotential surface passing through P.*

In the language of vector analysis, the attraction A is the **gradient** of the potential U.

The orthogonal trajectories of the family of equipotential surfaces $U = c$ are called **lines of force** since each of these curves has the property that the tangent at any one of its points has the direction of the attraction at that point. The differential equations of the lines of force are evidently

$$\frac{dx}{\frac{\partial U}{\partial x}} = \frac{dy}{\frac{\partial U}{\partial y}} = \frac{dz}{\frac{\partial U}{\partial z}}.$$

260. Exercises.

(1) For a mass spread uniformly over the surface of a sphere prove that, within the sphere, the potential is zero while, outside the sphere, it is the same as if the mass were concentrated at the center. Hence deduce the corresponding results for a homogeneous solid spherical shell.

(2) A mass is distributed uniformly along the arc of a parabola bounded by the latus rectum $4a$; show that at the focus the potential is $= 3.5245\ \kappa\rho''$ and the attraction is $= 1.8856\ \kappa\rho''/a$.

(3) Find the potential due to a homogeneous circular plate, of radius a, at a point P of its axis, at the distance x from the plate.

(4) Determine the equipotential surfaces for a straight homogeneous rod; comp. Art. 254, Ex. 4 and 5.

(5) For a mass distributed uniformly along the circumference of a circle, determine the potential at any point in the plane of the circle, and show that at a distance from the center equal to $\frac{2}{3}$ the radius it is $= 7.2418\ \kappa\rho''$.

(6) Show that a force-function exists when the resultant force is constant in magnitude and direction.

(7) Find the force-function in the case of a free particle moving under the action of the constant force of gravity (projectile *in vacuo*); determine the equipotential surfaces.

(8) Show the existence of a force-function when the direction of the resultant force is constantly perpendicular to a fixed plane, say the xy-plane, and its magnitude is a given function $f(z)$ of the distance z from the plane.

(9) Find the force-function, the equipotential surfaces, and the kinetic energy when the force is a function $f(r)$ of the perpendicular distance r from a fixed line, and is directed towards this line at right angles to it.

(10) Show the existence of a force-function for a central force, *i. e.* a force passing through a fixed point (x_0, y_0, z_0), if the force is a function of the distance r from this point. What are the level surfaces?

(11) Show that a force-function exists when a particle moves under the action of any number of such central forces as in Ex. (10).

3. Virtual work.

261. The importance of the potential in the theory of attraction and of the force-function for any conservative forces (Art. 255) is largely due to their connection with the idea of work.

The **work** W of a *constant* force F in a rectilinear displacement s of its point of application is defined as the product of the projection of F on s into s:

$$W = Fs \cos\psi,$$

where ψ is the angle between the vectors F and s. In other words, work is the dot-product (Art. 141) of force and displacement:

$$W = F \cdot s.$$

Thus, *e. g.* when a body of weight $F = mg$ slides down the greatest slope of a smooth plane inclined at the angle θ to the horizon, through a distance s, the work of the vertical force F is

$$Fs \cos(\tfrac{1}{2}\pi - \theta) = Fs \sin\theta = Fh,$$

where $h = s \sin\theta$ is the vertical height through which the body has descended.

It follows from the theory of projection (Art. 198) that *the work of a force is the sum of the works of its components.* Hence, if X, Y, Z are the rectangular components of F, x, y, z those of s, we have (comp. Art. 141)

$$W = Xx + Yy + Zz.$$

262. Work is not a vector, but a scalar quantity (Art. 259). If, in the definition of Art. 261, we take for ψ the lesser of the two angles made by the vectors F and s, the work is positive or negative according as ψ is $<$ or $> \frac{1}{2}\pi$

The dimensions of work are evidently ML^2T^{-2}.

The unit of work is the work of a unit force (poundal, dyne) through a unit distance (foot, centimeter). The unit of work in the F.P.S. system is called the **foot-poundal**; in the C.G.S. system, the **erg**. Thus, the erg is the amount of work done by a force of one dyne acting through a distance of one centimeter. These are the scientific units.

In the gravitation system where the pound, or the kilogram is taken as unit of force, the British unit of work is the **foot-pound,** while in the metric system it is customary to use the **kilogram-meter** as unit.

263. The numerical relations between these units are obtained as follows. Let x be the number of ergs in the foot-poundal, then (comp. Art. 175),

$$x \cdot \frac{\text{gm. cm.}^2}{\text{sec.}^2} = 1 \cdot \frac{\text{lb. ft.}^2}{\text{sec.}^2},$$

hence

$$x = \frac{\text{lb.}}{\text{gm.}} \cdot \left(\frac{\text{ft.}}{\text{cm.}}\right)^2 = 4.2141 \times 10^5;$$

i. e. 1 foot-poundal $= 4.2141 \times 10^5$ ergs, and 1 erg $= 2.3730 \times 10^{-6}$ foot-poundals.

Again, let x be the number of kilogram-meters in 1 foot-pound, then

$$x \text{ kg. m.} = 1 \text{ ft. lb.},$$

hence

$$x = \frac{\text{lb.}}{\text{kg.}} \cdot \frac{\text{ft.}}{\text{m.}} = 0.138\,257$$

i. e. 1 foot-pound $= 0.138\,257$ kilogram-meters.

Finally, 1 foot-pound $= g$ foot-poundals (Art. 179); hence 1 foot-pound $= 1.356 \times 10^7$ ergs, and 1 erg $= 7.3730 \times 10^{-8}$ foot-pounds, if $g = 981$.

264. Exercises.

(1) A *joule* being defined as 10^7 ergs, show that 1 foot-pound = 1.356 joules, and that 1 joule is about 3/4 foot-pound.

(2) Show that a kilogram-meter is nearly 10^8 ergs.

(3) What is the work done against gravity in raising 300 lbs. through a height of 25 ft.: (*a*) in foot-pounds, (*b*) in ergs?

(4) Find the work done against friction in moving a car weighing 3 tons through a distance of 50 yards on a level road, the coefficient of friction being 0.02.

(5) A mass of 12 lbs. slides down a smooth plane inclined at an angle of 30° to the horizon, through a distance of 25 ft.; what is the work done by gravity?

265. The work of a *variable* force F in a very small displacement $PP' = \delta s$ is defined (like that of a constant force in any displacement, Art. 261) as the product of δs into the projection $F \cos\psi$ of F (at P) on δs:

$$\delta W = F \cos\psi \, \delta s = F \cdot \delta s = X\delta x + Y\delta y + Z\delta z.$$

This expression is often called the **virtual work** of F in the **virtual displacement** δs, the term virtual and the letter δ meaning that the displacement is arbitrary and not necessarily the *actual* displacement along the path of the particle.

But it should be carefully observed that even if the displacement δs were taken along the actual path we do *not* in general have in the limit

$$\frac{dW}{ds} = F \cos\psi;$$

i. e. the s-component of the force is not necessarily an exact derivative.

The work done by the variable force F as the particle on which it acts is moved along an arbitrary curve from P_0 to any position P is written

$$W = \lim_{\delta s = 0} \Sigma F \cos\psi \, \delta s = \int_{P_0}^{P} F \cos\psi \, ds = \int_{P_0}^{P} F \cdot ds$$
$$= \int_{P_0}^{P} (X dx + Y dy + Z dz).$$

This integral can in general not be evaluated unless the path of the particle from P_0 to P is known; and it has in general

different values for different paths between these points.

But we have seen (Arts. 257, 258) that for a particle m in a field of Newtonian attraction the component of the resultant attraction in any direction s is the s-derivative of the potential: $A_s = dU/ds$. Hence, multiplying by m, we have in this case for the virtual work:

$$\delta W = mA_s\delta s = m\delta U.$$

It follows that the work done on the particle m by the Newtonian attraction, as it is moved from P_0 to P along *any* path, is

$$W = m\int_{P_0}^{P}\delta U = m(U - U_0),$$

where U_0 is the value of U at P_0. Hence *the work of attraction is independent of the path*; it is m times the *difference of potential* at P and P_0; it is zero in any *closed* path.

More generally, whenever the force F is *conservative* (Art. 255) so that it possesses a one-valued force-function, *i. e.* a function $U(x, y, z)$ such that $\partial U/\partial x$, $\partial U/\partial y$, $\partial U/\partial z$ are the rectangular components of F, the projection of F on any direction s will be the s-derivative of U, and hence the work of F is independent of the path.

266. For a *particle in equilibrium*, since the resultant force F is zero, it follows that the virtual work $\delta W = F\cos\psi\,\delta s$ is zero whatever the displacement δs. And conversely, if the virtual work is zero whatever δs, or more exactly, if the virtual work is small of an order higher than that of δs for every sufficiently small δs, the resultant force F must be zero, *i. e.* the particle is in equilibrium.

The virtual work is zero for every δs if it is zero for any three non-complanar displacements.

Using rectangular co-ordinates we have $\delta W = X\delta x +$

$Y\delta y + Z\delta z$; hence $\delta W = 0$ when $X = 0$, $Y = 0$, $Z = 0$; and conversely, if $\delta W = 0$ for a virtual, *i. e.* arbitrary, displacement, we have owing to the independence of δx, δy, δz: $\delta W = 0$.

The proposition that *the vanishing of the virtual work* (apart from terms of a higher order) *is a necessary and sufficient condition of equilibrium for a particle* is known as the **principle of virtual work** for the particle.

267. In the particular case of a particle in a field of *conservative* forces whose force-function is U, the condition of equilibrium assumes the form

$$\frac{dU}{ds} = 0$$

for any ds· or, with reference to rectangular axes:

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0.$$

Now these are necessary conditions for a maximum or minimum of U. Hence the positions of equilibrium of a particle under conservative forces are found by determining the maxima and minima of the force-function or potential.

It can be shown that a minimum of U corresponds to stable, a maximum to unstable, equilibrium.

268. The principle of virtual work, proved above only for the single free particle, has a far wider field of application. It can be shown that *for any system of particles or rigid bodies, subject to any constraints, expressible by equations* (not inequalities) *and not involving friction, the vanishing of the virtual work* (apart from terms of higher order) *for any displacement compatible with the constraints is a necessary and sufficient condition of equilibrium.*

If in the expression of the virtual work $\delta W = F \cos\psi\, \delta s$ we replace δs by $(\delta s/\delta t)\delta t$ we can regard $\delta s/\delta t$ as a *velocity*. This is the reason why the principle of virtual work is often called the *principle of virtual velocities*.

PART III: KINETICS.

CHAPTER XIII.

MOTION OF A FREE PARTICLE.

1. The equations of motion.

269. Let a particle of mass m be acted upon by any number of forces; as these forces are concurrent they are equivalent to a single resultant R (Art. 190). The definition of force (Art. 171) then gives for the acceleration j the fundamental **equation of motion**

$$mj = R. \tag{1}$$

The mass m being regarded as a positive constant the equation shows that the vectors j and R have the same direction and sense.

The vector equation (1) assumes various forms according to the method selected for resolving j and R into components.

If the motion be referred to fixed rectangular axes, (1) is replaced by the three equations (Art. 53):

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z, \tag{2}$$

X, Y, Z being the components of R along Ox, Oy, Oz.

If polar co-ordinates r, θ, φ are used we have (Art. 56, Ex. 9):

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2) &= R_r, \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2) &= R_\theta, \\ m(r\sin\theta\,\ddot{\varphi} + 2\sin\theta\,\dot{r}\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi}) &= R_\phi, \end{aligned} \tag{3}$$

where R_r, R_θ, R_ϕ are the components of R along the radius vector, at right angles to the radius vector in the meridian plane, and at right angles to this plane.

Finally, resolving along the tangent, normal, and binormal to the path we have (Art. 51):

$$m\dot{v} = m\ddot{s} = R_t, \quad m\frac{v^2}{\rho} = R_n, \quad 0 = R_b. \tag{4}$$

In the case of *plane motion* the equations (2), (3), (4) reduce to the first two, with $\dot{\varphi} = 0$ in (3); in the case of *rectilinear motion* the first equation of (2) or (4) suffices.

270. If the components X, Y, Z were given as functions of the time t alone, each of the three equations (2) could be integrated separately. In general, however, these components will be functions of the co-ordinates, and perhaps also of the velocity and of the time. No general rules can be given for integrating the equations in this case. By combining the equations (2) in such a way as to produce exact derivatives in the resulting equation, it is sometimes possible to effect an integration. Two methods of this kind have been indicated for the case of two dimensions in a particular example in Kinematics, Arts. 102–104. We now proceed to study these *principles* from a more general point of view, and to point out the physical meaning of the expressions involved.

271. The Principle of Kinetic Energy and Work. Let us combine the equations of motion (2) by multiplying them by $\dot{x}$, $\dot{y}$, $\dot{z}$, respectively, and then adding. As $\dot{x}\ddot{x}$ is the time derivative of $\frac{1}{2}\dot{x}^2$, the left-hand member of the resulting equation will be the t-derivative of $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mv^2$, *i. e.* of the kinetic energy of the particle (Art. 181). We find therefore

$$\frac{d}{dt}\,\tfrac{1}{2}mv^2 = X\dot{x} + Y\dot{y} + Z\dot{z}.$$

Hence, integrating from any point P_0 of the path where $v = v_0$ to any point P we obtain:

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{P_0}^{P}(Xdx + Ydy + Zdz). \qquad (5)$$

The left-hand member represents the increase in the kinetic energy of the particle; the right-hand member represents the work done by the resultant force R, since its work is equal to the sum of the works of its components X, Y, Z (Art. 261). Equation (5) states, therefore, that *the amount by which the kinetic energy increases, as the particle passes from P_0 to P, is equal to the work done by the resultant force R on the particle.*

272. The principle of kinetic energy and work can also be deduced from the former of the two equations (4). Multiplying this equation by $v = ds/dt$, we have

$$\frac{d(\frac{1}{2}mv^2)}{dt} = R_t\frac{ds}{dt} = R\cos\psi\frac{ds}{dt};$$

hence, integrating as in Art. 271:

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{s_0}^{s}R\cos\psi\, ds, \qquad (5')$$

where ψ is the angle made by the force R with the tangent to the path.

The integrand in (5) or (5′), *i. e.* the expression

$$R\cos\psi\, ds = R\cdot ds = Xdx + Ydy + Zdz,$$

is called the *elementary work.* It is the value of the virtual work (Art. 265) when the displacement δs is taken infinitesimal and along the actual path.

As explained in Art. 265, the evaluation of the work integral in general requires a knowledge of the path. As in many problems the path is not known beforehand, but is

one of the things to be determined, it is very important to notice that *in the case of conservative forces* (Art. 255) *the work integral has a value independent of the path* (Art. 265). In this case, denoting the force-function, or potential, by U, we have

$$\int_{P_0}^{P}(Xdx + Ydy + Zdz) = \int_{P_0}^{P} dU = U - U_0,$$

so that the equation (5) or (5′) becomes

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = U - U_0. \tag{6}$$

Hence in the case of conservative forces the principle of kinetic energy and work at once gives a *first integral of the equations of motion.*

273. The negative of the force-function, say

$$V = -U,$$

is called the **potential energy.** If this quantity be introduced and the kinetic energy be denoted by T, the equation (6) assumes the form

$$T + V = T_0 + V_0, \tag{6′}$$

which expresses the **principle of the conservation of energy** for a particle: *the total energy, i. e. the sum of the kinetic and potential energies, remains constant throughout the motion if the forces are conservative.* In other words, whatever is gained in kinetic energy is lost in potential energy, and vice versa.

274. The physical idea to which the term *potential energy* is due can perhaps best be explained by considering the Newtonian attraction between two particles m, m'. We think of the attracting particle m' as generating a field. Wherever in this field a particle m be placed (say, with zero velocity), it will become subject to the attraction A of m' and move toward m' with increasing velocity, thus acquiring kinetic energy; at the same time the force A does an amount of work on m which is exactly equivalent to the kinetic energy gained by m. It follows that,

the farther away from m' the particle m is placed, initially, the greater will be the amount of work that m' can do upon it. It is this "potentiality" for doing work, due to the distance of m from m', which is denoted as *energy of position*, or *potential energy*. The equation (6), or the equation (6′) which differs from (6) merely in notation, shows that *what the particle m in moving toward m′ gains in kinetic energy it loses in potential energy* so that the sum of kinetic and potential energy always remains constant.

275. The conditions for the existence of a force-function are (Art. 255):

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}.$$

Differentiating the second equation with respect to z, the third with respect to y we find

$$\frac{\partial Y}{\partial z} = \frac{\partial^2 U}{\partial z \partial y}, \quad \frac{\partial Z}{\partial y} = \frac{\partial^2 U}{\partial y \partial z},$$

whence $\partial Y/\partial z = \partial Z/\partial y$. Proceeding in the same way with the other two pairs of equations we find:

$$\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}, \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}. \tag{7}$$

These relations which are necessary and sufficient for the existence of a force-function U furnish a simple criterion for recognizing whether the given forces are conservative.

276. The principle of the conservation of energy, *i. e.* of the constancy of the sum of kinetic and potential energy, has been proved mathematically in the preceding articles for a very particular case, viz. for the motion of a particle under conservative forces.

By a generalization as bold and far-reaching as was Newton's extension of the property of mutual attraction to all matter (Art. 244), modern physics has been led to the assumption that *work and energy are quantities which can*

never be destroyed, but can be transformed in a variety of ways. This assumption, the **general principle of the conservation of energy,** while fully borne out so far by the results deduced from it, is of course not capable of mathematical proof. Indeed, it may be said that in defining the various forms of energy, such as heat, chemical energy, radio-activity, etc., the *definitions* are so formulated as to conform to this principle; it has always been found possible to do this. The general principle of the conservation of energy cannot be fully discussed here, since this would require a study of all the forms of energy known to physics.

277. In its application to machines, the principle states that the *total work* W supplied to a machine in a given time by the agent, or motor, driving it (such as animal force, the expansive force of steam, the pressure of the wind, the impact of water, etc.) is equal to the sum of the *useful work* W_u, done by the machine in the same time and the so-called *lost*, or *wasteful, work* W_w spent in overcoming friction and other passive resistances of the machine:

$$W = W_u + W_w.$$

While W and W_u can be determined with considerable accuracy, it is difficult to determine W_w directly with equal precision; but it is found that the more accurately in any given machine W_w is determined, the more nearly will the above equation be found satisfied. This serves as a verification of the principle of the conservation of energy in its application to machines. The ratio W_u/W of the useful work to the total work is called the **efficiency** of the machine. The term *modulus* is sometimes used for efficiency.

278. *The time-rate at which work is performed by a force* has received a special name, **power** or **activity.** The source from which the force for doing useful work is derived is commonly called the *agent*, or *motor;* and it is customary to speak of the power of an agent, this meaning the rate at which the agent is capable of supplying work.

The *dimensions* of power are evidently ML^2T^{-3}. The *unit of power* is the power of an agent that does unit work in unit time. Hence,

in the scientific system, it is the power of an agent doing one erg per second in the C.G.S. system, and one foot-poundal per second in the F.P.S. system. As, however, the idea of power is of importance mainly in engineering practice, power is usually measured in gravitation units. In this case, the unit of power is the power of an agent doing one foot-pound per second in the F.P.S. system, and one kilogram-meter in the metric system.

A larger unit is frequently found more convenient. For this reason, the name **horse-power** (H.P.) is given to the power of doing 550 foot-pounds of work per second, or $550 \times 60 = 33{,}000$ foot-pounds per minute.

279. The principle of angular momentum or of areas. By multiplying the first of the equations of motion (2), Art. 269, by y, the second by x, and then subtracting the first from the second we obtain the equation

$$m(x\ddot{y} - y\ddot{x}) = xY - yX,$$

or since the left-hand member is the time-derivative of $m(x\dot{y} - y\dot{x})$:

$$\frac{d}{dt}m(x\dot{y} - y\dot{x}) = xY - yX.$$

Here the right-hand member is the moment of the resultant force R about the axis Oz (Art. 229) while, on the left, the quantity $x \cdot m\dot{y} - y \cdot m\dot{x}$ is the moment about the same axis of the *momentum* mv whose components are $m\dot{x}$, $m\dot{y}$, $m\dot{z}$ (Art. 168). This *moment of momentum* $m(x\dot{y} - y\dot{x})$ is also called **angular momentum.**

As any line might have been chosen as axis Oz, our equation expresses the proposition: *In the motion of a particle, the time-rate of change of the angular momentum about any line is equal to the moment of the resultant force about the same line.*

Applying this result to each of the axes of reference we find:

$$\frac{d}{dt}m(y\dot{z} - z\dot{y}) = yZ - zY,$$
$$\frac{d}{dt}m(z\dot{x} - x\dot{z}) = zX - xZ, \tag{8}$$
$$\frac{d}{dt}m(x\dot{y} - y\dot{x}) = xY - yX.$$

These equations express the *principle of angular momentum or of areas.*

280. To interpret these equations geometrically consider first the right-hand members which are the moments of the resultant force R about the axes. The vector $PA = R$ (Fig. 70) forms with the origin O a triangle whose area is

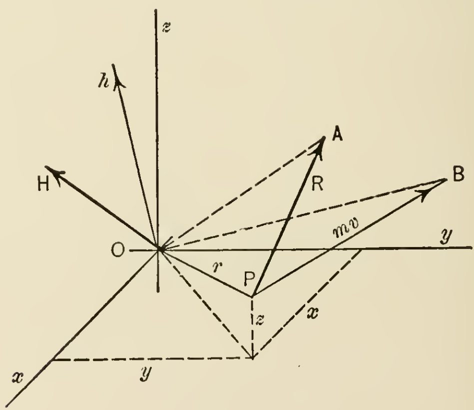

Fig. 70.

one-half the moment of R about O; let us represent this moment, which is the cross-product of the radius vector $OP = r$ and the force-vector $PA = R$, by a vector H perpendicular to the plane of the triangle OPA (comp. Arts. 199 and 119):

$$H = r \times R,$$

the length of this vector H being equal to twice the area OPA. The projection of the triangle OPA on the xy-plane has the area $\frac{1}{2}(xY - yX)$ since the vertices of this projection have the co-ordinates $(0, 0)$, (x, y), and $(x + X, y + Y)$; hence the right-hand members of (8) are the components H_x, H_y, H_z of the vector H.

Next consider in the same way the momentum-vector $mv = PB$; it forms with O a triangle OPB whose area is one-half the moment of momentum about O. We can represent this moment of momentum, or angular momentum, by a vector h, perpendicular to the plane OPB, and of a length equal to twice the area of the triangle OPB; the vector h is then the cross-product of $r = OP$ and $mv = PB$:

$$h = r \times mv.$$

The components of angular momentum $m(y\dot{z} - z\dot{y})$, $m(z\dot{x} - x\dot{z})$, $m(x\dot{y} - y\dot{x})$ are the components h_x, h_y, h_z of the vector h.

The equations (8) can therefore be written in the form

$$\frac{dh_x}{dt} = H_x, \quad \frac{dh_y}{dt} = H_y, \quad \frac{dh_z}{dt} = H_z; \tag{8'}$$

and these equations can be combined into the single vector equation

$$\frac{dh}{dt} = H,$$

which means that the geometrical increment of the vector h, divided by Δt, gives in the limit the vector H; *i. e. the* (geometrical) *time-rate of change of the angular-momentum vector is equal to the moment-vector of the resultant force.*

281. If instead of the momentum-vector mv we consider the velocity-vector v, its moment about O would be represented by the vector $(1/m)h$, whose components are $y\dot{z} - z\dot{y}$,

$z\dot{x} - x\dot{z}$, $x\dot{y} - y\dot{x}$. These quantities are (Art. 47) equal to twice the sectorial velocities about the axes while the vector $(1/m)h$ represents twice the sectorial velocity of the particle about O. This explains the name *principle of areas*.

282. If, in particular, the resultant force R is *central, i. e.* such as to pass always through a fixed point, then, for this point as origin, the right-hand members of the equations (8) are zero, and we find at once the *first integrals* of the equations of motion (2):

$$m(y\dot{z} - z\dot{y}) = h_1, \quad m(z\dot{x} - x\dot{z}) = h_2, \quad m(x\dot{y} - y\dot{x}) = h_3, \tag{9}$$

where h_1, h_2, h_3 are constants.

Thus, in the motion of a particle in the field of a central force, the angular momentum, and hence the sectorial velocity, about any axis through the center is constant.

If the resultant force always intersects a fixed line, the angular momentum, and hence the sectorial velocity, about this line as axis remains constant.

These propositions are often referred to as the *principle of the conservation of angular momentum or of areas*.

It may be noted that the equations (9), multiplied by x, y, z and added give,

$$h_1x + h_2y + h_3z = 0;$$

this shows that the particle moves in a plane passing through the center of force, as is otherwise evident.

283. Exercise.

In the case of plane motion, if the plane be taken as the xy-plane, the principle of areas is expressed by the third of the equations (8). If the perpendicular from the origin O to the tangent at P be denoted by p (comp. Art. 100), this equation can be written in the form $d(mpv)/dt = xY - yX$. Show that the two terms $mpdv/dt$ and $mvdp/dt$ of the left-hand member represent the moments of the tangential and normal components of the resultant force R, respectively.

2. Examples of rectilinear motion.

284. Free Oscillations. As an example of *rectilinear motion* consider the motion of a particle of mass m under a force directly proportional to the distance $OP = s$ of the particle from a fixed point O. If the force is attractive, *i. e.* directed toward the point O and if the initial velocity passes through O or is zero so that the motion is rectilinear, the single equation of motion is

$$m\ddot{s} = -m\kappa^2 s, \tag{10}$$

and the motion (see Arts. 26, 27, 71) is a simple harmonic oscillation or vibration about the point O as *center*. This point O, at which the force $R = -m\kappa^2 s$ is zero, is therefore a position of equilibrium for the particle.

The potential energy V due to the force $R = -m\kappa^2 s$ is, by Art. 273,

$$V = -\int R ds = m\kappa^2 \int s ds = \tfrac{1}{2} m\kappa^2 s^2 + C.$$

Hence the principle of the conservation of energy gives

$$v^2 + \kappa^2 s^2 = \text{const.}$$

If the initial velocity be zero for $s = s_0$, we have

$$v = \mp \kappa \sqrt{s_0{}^2 - s^2}.$$

285. As in the applications the moving particle m is generally subject to the constant force of gravity, it is important to notice that the introduction of a constant force F along the line of motion does not essentially change the character of the motion. For, the equation of motion

$$m\ddot{s} = -m\kappa^2 s + F = -m\kappa^2 \left(s - \frac{F}{m\kappa^2} \right)$$

reduces, with $s - F/m\kappa^2 = x$, to

$$m\ddot{x} = -m\kappa^2 x,$$

which agrees in form with (10). The only change in the results is that

the *center* of the oscillations, *i. e.* the position of equilibrium of the particle m, is not the point O, but a point at the distance $e = F/m\kappa^2$ from O.

286. Forces proportional to a distance, or length, are directly observed in the stretching of so-called *elastic* materials. Thus, a homogeneous straight steel wire when suspended vertically from one end and weighted at the other end is found to stretch; and careful measurements have shown that the extension, or change of length, is directly proportional to the weight applied (the weight of the wire itself being assumed, for the sake of simplicity, as very small in comparison with the load applied). Conversely, the *tension*, or *elastic stress*, of the wire is proportional to the *extension* produced. Moreover, when the weight is removed the wire is found to contract to its original length.

This physical law, known as *Hooke's law of elastic stress*, holds only within certain limits. If the weight exceeds a certain limiting value, the extension is no longer proportional to the weight, and after removing the weight, the wire does not regain its original length, but is found to have acquired a *permanent set*, or lengthening; it is said in this case that the elastic limits have been exceeded.

Materials for which Hooke's law holds exactly within certain limits of tension and extension are called *perfectly elastic*. Strictly speaking, such materials probably do not exist; but many materials follow Hooke's law very closely within proper limits. Thus, elastic strings, such as rubber bands, and spiral steel springs show these phenomena very clearly on account of the large extensions allowable within the elastic limits.

287. The elastic constant $m\kappa^2$. Let an elastic string whose natural length is l assume the length $l + x$ when the tension is F, so that according to Hooke's law,

$$F = -m\kappa^2 x.$$

To determine the factor of proportionality $m\kappa^2$ for a given string, we may observe the length l_1 assumed by the string under a known tension, *e. g.* the tension $-m_1 g$ produced by suspending a given mass m_1 from the string (the weight of the string itself being neglected).

We then have

$$-m_1 g = -m\kappa^2(l_1 - l),$$

whence

$$m\kappa^2 = \frac{m_1 g}{l_1 - l},$$

and

$$F = -\frac{m_1 g}{l_1 - l} x.$$

288. Let the same string be placed on a smooth horizontal table, one end being fixed at a point O (Fig. 71), while a particle of mass m is

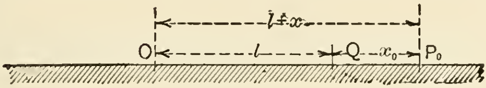

Fig. 71.

attached to the other end. Stretch the string to a length $OP_0 = l + x_0$ (within the limits of elasticity) and let go; the particle m will move under the action of the tension F alone, its weight being balanced by the reaction of the table. The equation of motion is

$$m\ddot{x} = -\frac{m_1 g}{l_1 - l} x,$$

the distance $QP = x$ being counted from the point Q at the distance $OQ = l$ from the fixed point O. Putting again (Art. 287)

$$\kappa = \sqrt{\frac{m_1 g}{m(l_1 - l)}},$$

and integrating, we find

$$x = c_1 \cos\kappa t + c_2 \sin\kappa t,$$

whence

$$v = \dot{x} = -\kappa c_1 \sin\kappa t + \kappa c_2 \cos\kappa t.$$

As $x = x_0$ and $v = 0$ for $t = 0$, we have $c_1 = x_0$, $c_2 = 0$; hence

$$x = x_0 \cos\kappa t, \quad v = -\kappa x_0 \sin\kappa t.$$

It should be noticed that these equations hold only as long as the string is actually stretched, *i. e.* as long as $x > 0$. The subsequent motion is, however, easily determined from the velocity for $x = 0$.

289. It was assumed, in the preceding article, that the particle m is let go from its initial position P_0 with zero velocity. This can be brought about by pulling the particle from Q to P_0 with a gradually increasing force which at any point P is just equal and opposite to the

corresponding elastic tension, or *stress*, $P = m\kappa^2 x$. The work thus done against the tension, *i. e.* in stretching or *straining* the string, is stored in the particle m as potential energy, or *strain energy*, V. To find its amount, observe that, as the particle m is pulled through the short distance Δx, the work of the force is $= m\kappa^2 x\Delta x$; this being the potential energy ΔV gained in the distance Δx, we have $\Delta V = m\kappa^2 x\Delta x$; hence

$$V_0 = \int_0^{x_0} m\kappa^2 x dx = \tfrac{1}{2}m\kappa^2 x_0^2.$$

Thus, in the initial position P_0 the particle m possesses this potential energy, but no kinetic energy. During its motion from P_0 to Q, the particle gains kinetic energy and loses potential energy. At any intermediate point P, for which $QP = x$, the kinetic energy is $T = \frac{1}{2}mv^2$, while the potential energy is $V = \frac{1}{2}m\kappa^2 x^2$. By the principle of the conservation of energy (Art. 273), the sum of these two quantities, the so-called *total energy*, E, remains constant as long as no other forces besides the elastic stress act on the particle:

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}m\kappa^2 x^2 = \text{const.}$$

The value of the constant is $= \frac{1}{2}m\kappa^2 x_0^2$, since this is the total energy at P_0; hence,

$$v^2 + \kappa^2 x^2 = \kappa^2 x_0^2.$$

(Comp. Art. 284). This relation also follows from the values of x and v given in Art. 288, upon eliminating t.

When the particle arrives at the position of equilibrium Q, the potential, or strain, energy has been consumed, having been converted completely into kinetic energy.

290. Exercises.

(1) In the problem of Art. 288 let the string be a rubber band whose natural length of 1 ft. is increased 3 in. when a weight of 4 oz. is suspended from it; determine the motion of a 1-oz. particle attached to one end, the band being initially stretched to a length of 1½ ft.; find (*a*) the greatest tension of the band, (*b*) the greatest velocity of the particle, (*c*) the period, (*d*) the work done by the tension in a quarter oscillation.

(2) Discuss the effect of friction, of coefficient μ, in the problem of Art. 288.

(3) The length $OQ = l$ of an elastic string is increased to $OQ_1 = l_1 =$

$l + e$ if a mass m is suspended from its lower end, the upper end O being fixed (Fig. 72). The mass m is pulled down to the distance $Q_1P_0 = x_0$ from the position of equilibrium Q_1 and then released. Prove the following results: With Q_1 as origin the equation of motion of m is

$$\ddot{x} = -\kappa^2 x, \text{ where } \kappa = \sqrt{\frac{g}{e}},$$

whence

$$x = x_0 \cos\kappa t,\ v = -\kappa x_0 \sin\kappa t.$$

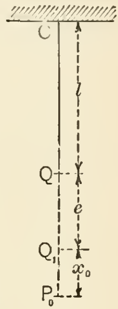

Fig. 72.

If $x_0 < e$, the tension never vanishes, and m performs isochronous oscillations of period $2\pi\sqrt{e/g}$, the period being the same as for the small oscillations of a pendulum of length e. If $x_0 > e$, the tension vanishes for $x = -e$, *i. e.* at Q; the velocity at this point is $v_1 = -\kappa\sqrt{x_0^2 - e^2}$, and the particle rises to the height $h = (x_0^2 - e^2)/2e$ above Q. The total time of one up and down motion is

$$2\sqrt{e/g}[\tfrac{1}{2}\pi + \sin^{-1}(e/x_0) + \sqrt{(x_0/e)^2 - 1}].$$

(4) How is the motion of Ex. (3) modified if the elastic string be replaced by a spiral spring suspended vertically from one end? Assume the resistance of the spring to compression equal to its resistance to extension.

(5) The particle in Ex. (3) is let fall from a height h above Q; determine the greatest extension of the string.

(6) An elastic string whose natural length is l is suspended from a fixed point. A mass m_1 attached to its lower end stretches it to a length l_1; another mass m_2 stretches it to a length l_2. If both these masses be attached and then the mass m_2 be cut off, what will be the motion of m_1?

(7) If a straight smooth hole be bored through the earth, connecting any two points A, B on the surface, in what time would a particle slide from A to B? The attraction in the interior is directly proportional to the distance from the center of the earth.

291. Resistance of a Medium. It is known from observation that the velocity v of a rigid body moving in a liquid or gas is continually diminished, the medium apparently exert-

ing on the body a retarding force which is called the *resistance of the medium*. This force F is found to be roughly proportional to the density ρ of the medium, the greatest cross-section A of the body (at right angles to the velocity v), and generally, at least for large velocities, to the square of the velocity v:

$$F = k\rho Av^2,$$

where k is a coefficient depending on the shape and physical condition of the surface of the body.

This expression for the resistance F can be made plausible by the following consideration. As the body moves through the medium, say with constant velocity v, it imparts this velocity to the particles of the medium it meets. The portion of the medium so affected in the unit of time can be regarded as a cylinder of cross-section A and length v, and hence of mass ρAv. To increase the velocity of this mass from 0 to v in the unit of time requires, by equation (5) of Art. 171, a force

$$\frac{\rho Av \cdot v}{1} = \rho Av^2.$$

The retarding force of the medium must be equal and opposite to this force multiplied by a coefficient k to take into account various disturbing influences.

For small velocities, however, the resistance can be assumed proportional to the velocity, $F = kv$, the coefficient k to be determined by experiment.

The above consideration is only a very rough approximation. Thus the particles of the medium are not simply given the velocity v in the direction of motion; they are partly pushed aside and move in curves backwards, causing often whirls or eddies alongside and behind the body. If the medium is a gas, it is compressed in front, and rarefied behind the body; indeed, when the velocity is great (greater than that of sound in the gas), a vacuum will be formed behind the body. Moreover, a layer of the medium adheres to and moves with the body, thus increasing the cross-section. It is therefore often found necessary to assume a more general expression for the resistance; and this is, in ballistics, generally written in the form

$$F = \kappa\rho Av^2 f(v).$$

The careful experiments that have been made to determine the resistance offered by the air to the motion of projectiles have shown that for velocities up to about 250 meters per second, as well as for velocities above 420 m./sec., $f(v)$ can be regarded as constant, *i. e.* the resistance is proportional to the square of the velocity. But for velocities between 250 and 420 m./sec., *i. e.* in the vicinity of the velocity of sound in air (330–340 m./sec.), the law of resistance is more complicated.

292. Falling Body in Resisting Medium. Assuming the resistance proportional to the square of the velocity, the equation of motion for a body falling (without rotating) in a medium of constant density is

$$m\frac{d^2s}{dt^2} \equiv m\frac{dv}{dt} = mg - mkv^2,$$

where k is a positive constant. To simplify the resulting formulæ, put

$$k = \frac{\mu^2}{g};$$

then the separation of the variables v and t gives

$$dt = \frac{gdv}{g^2 - \mu^2 v^2},$$

whence

$$t = \frac{1}{2\mu}\log\frac{g + \mu v}{g - \mu v},$$

the constant of integration being zero if the initial velocity is zero. Solving for v, we have

$$v = \frac{g}{\mu}\cdot\frac{e^{\mu t} - e^{-\mu t}}{e^{\mu t} + e^{-\mu t}} = \frac{g}{\mu}\tanh \mu t.$$

Writing ds/dt for v and integrating again, we find, since $s = 0$ for $t = 0$,

$$s = \frac{g}{\mu^2}\log\tfrac{1}{2}\left(e^{\mu t} + e^{-\mu t}\right) = \frac{g}{\mu^2}\log\cosh \mu t.$$

The relation between v and s can be obtained by eliminating t between the expressions for v and s, or more conveniently by eliminating t from the original differential equation by means of the relation

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

This gives

$$ds = \frac{gvdv}{g^2 - \mu^2 v^2},$$

whence, with $v = 0$ for $s = 0$,

$$s = \frac{g}{2\mu^2}\log\frac{g^2}{g^2 - \mu^2 v^2}.$$

293. Exercises.

(1) Show that, as t increases, the motion considered in Art. 292 approaches more and more a state of uniform motion without ever reaching it.

(2) Determine the motion of a body projected vertically upward in the air with given initial velocity v_0, the resistance of the air being proportional to the square of the velocity.

(3) In Ex. (2) find the whole time of ascent and the height reached by the particle.

(4) Show that, owing to the resistance of the air, a body projected vertically upward returns to the starting point with a velocity less than the initial velocity of projection.

(5) A ball, 6 in. in diameter, falls from a height of 300 ft.; find how much its final velocity is diminished by the resistance of the air, if $k = 0.00090$.

(6) Determine the rectilinear motion of a body in a medium whose resistance is proportional to the velocity, when no other forces act on it.

(7) A body falls from rest in a medium whose resistance is proportional to the velocity; find v and s in terms of t, v in terms of s.

294. Damped Oscillations. Let a particle of mass m be attracted by a fixed center O, with a force proportional to the distance from O, and move in a medium whose resistance

is proportional to the velocity. If the initial velocity be directed through O (or be zero), the motion will be rectilinear, and the equation of motion is

$$m\frac{d^2s}{dt^2} = -m\kappa^2 s - mkv,$$

or, putting $k = 2\lambda$,

$$\frac{d^2s}{dt^2} + 2\lambda\frac{ds}{dt} + \kappa^2 s = 0. \tag{11}$$

This is a homogeneous linear differential equation of the second order with constant coefficients, which can be integrated by a well-known process. The roots of the auxiliary equation,

$$-\lambda \pm \sqrt{\lambda^2 - \kappa^2},$$

are real or imaginary according as $\lambda > \kappa$, or $\lambda < \kappa$. The limiting cases $\lambda = \kappa$, $\lambda = 0$, $\kappa = 0$, also deserve special mention.

(*a*) If $\lambda > \kappa$, the roots are real and different, and as λ is positive, both roots are negative; denoting them by $-a$ and $-b$, so that a and b are positive constants, and $b > a$, the general solution is

$$s = c_1 e^{-at} + c_2 e^{-bt}.$$

As the force has a finite value at the center O, we can take $s = 0$, $v = v_0$ for $t = 0$ as initial conditions. This gives

$$s = \frac{v_0}{b-a}(e^{-at} - e^{-bt}), \quad v = \frac{v_0}{b-a}(be^{-bt} - ae^{-at}).$$

The velocity reduces to zero at the time

$$t_1 = \frac{1}{b-a}\log\frac{b}{a}.$$

As a and b are positive and $b > a$, s has always the sign of v_0, *i. e.* the particle remains always on the same side of O; it reaches its elongation at the time t_1, for which v vanishes, and then approaches the point O asymptotically.

Hence, in this case, the damping effect of the medium is sufficiently great to prevent actual oscillations. Such motions are sometimes called **aperiodic.**

(*b*) If $\lambda = \kappa$, the roots are real and equal, viz. $= -\lambda$, and the general solution is

$$s = (c_1 + c_2 t)e^{-\lambda t}.$$

With $s = 0$, $v = v_0$ for $t = 0$, we find

$$s = v_0 t e^{-\lambda t}, \quad v = v_0(1 - \lambda t)e^{-\lambda t}.$$

The velocity vanishes for $t_1 = 1/\lambda$, and then only. The nature of the motion is essentially the same as in the previous case.

(*c*) If $\lambda < \kappa$, the roots are complex, say $= -\alpha \pm \beta i$, where α and β are positive constants. The general solution

$$s = e^{-\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

gives with $s = 0$, $v = v_0$ for $t = 0$:

$$s = \frac{v_0}{\beta} e^{-\alpha t} \sin\beta t, \quad v = \frac{v_0}{\beta} e^{-\alpha t}(\beta \cos\beta t - \alpha \sin\beta t).$$

Here v vanishes whenever $\tan\beta t = \beta/\alpha = \sqrt{(\kappa/\lambda)^2 - 1}$; s vanishes (*i. e.* the particle passes through O) whenever t is an integral multiple of π/β; s has an infinite number of maxima and minima whose absolute values rapidly diminish.

The resistance of the medium, while not sufficient to extinguish the oscillations, continually shortens their amplitude; this is the typical case of *damped oscillations.*

(*d*) If $\lambda = 0$, the roots are purely imaginary, viz. $= \pm \kappa i$. In this case, the second term in equation (11) is zero; there

is no damping effect, and we have the case of *free oscillations* (see Arts. 284–290).

(*e*) If $\kappa = 0$, one of the roots is zero, the other is $= -2\lambda$. The attracting (or elastic) force being zero, we have the case of Ex. (6), Art. 293.

295. As shown in Arts. 273, 274, the *principle of the conservation of energy* holds for the *free* oscillations of a particle (under a force proportional to the distance). In the case of *damped* oscillations (Art. 294), this principle, in the restricted sense in which it has been proved so far, is not applicable, the resistance of the medium not being given as a function of the distance s. The total energy $E = T + V$ of the particle, or rather the energy stored in the system formed by the spring with the particle attached (in the example used above), diminishes in the course of time because the spring has to do work against the resistance of the medium, thus transferring part of its energy to the medium (setting it in motion, heating it, etc.). Thus, in a generalized meaning, the principle of the conservation of energy can be said to hold for the larger system, formed by the spring, together with the medium (see Art. 276).

The rate at which the total energy E diminishes with the *time* is here proportional to the *square* of the velocity:

$$\frac{dE}{dt} = -2m\lambda v^2;$$

for, substituting for E its value $E = T + V = \frac{1}{2}mv^2 + \frac{1}{2}mk^2s^2$ (Art. 289) and reducing, we find the equation of motion (11).

The *space*-rate of change of the total energy E is proportional to the velocity, and is nothing else but the resistance of the medium:

$$\frac{dE}{ds} = -2\,m\lambda v;$$

for we have

$$\frac{dE}{dt} = \frac{dE}{ds}\frac{ds}{dt} = v\frac{dE}{ds}.$$

296. Forced Oscillations. In the case of free simple harmonic oscillations, while the force regarded as a function of

the distance s is directly proportional to s, the same force regarded as a function of the time is of the form

$$R = -m\kappa^2 s_0 \cos\kappa t,$$

since $s = s_0 \cos\kappa t$. Conversely, a particle acted upon by a single force $R = mk \cos\mu t$, or $R = mk \sin\mu t$, directed toward a fixed center O, will, if the initial velocity passes through O, have a simple harmonic motion.

Suppose that such a force in the line of motion be superimposed in the case of Art. 294 so that the equation of motion becomes

$$m\frac{d^2s}{dt^2} = -m\kappa^2 s - 2m\lambda v + mk \cos\mu t,$$

or

$$\frac{d^2s}{dt^2} + 2\lambda\frac{ds}{dt} + \kappa^2 s = k \sin\mu t. \tag{12}$$

The particle is then said to be subject to *forced oscillations*. For a particle suspended from a spiral spring this could be realized by subjecting the point of suspension to a vertical simple harmonic motion of amplitude k and period $2\pi/\mu$.

The non-homogeneous linear differential equation (12) with constant coefficients can be integrated by well-known methods.

297. Exercises.

(1) With $\mu = 2$, $v_0 = 4$, sketch the curves representing s as a function of t in the five cases of Art. 294; take (*a*) $\lambda = 3$, (*b*) $\lambda = 2$, (*c*) $\lambda = \frac{1}{4}$, (*e*) $\lambda = 2$.

(2) Compare the cases (*c*) and (*d*) of Art. 294; show that the oscillations in a resisting medium are isochronous, but of greater period than *in vacuo*. The ratio of the amplitude at any time to the initial amplitude is called the *damping ratio;* show that the logarithm of this ratio, the so-called *logarithmic decrement*, is proportional to the time.

(3) Derive the equation of motion in the case of free oscillations from the principle of the conservation of energy.

(4) Integrate and discuss the equation $\ddot{s} + \kappa^2 s = a \sin\mu t$; show that the amplitude of the forced oscillation becomes very large if the periods of the free and forced oscillations are nearly equal. Discuss the limiting case when $\mu = \kappa$.

(5) Integrate (12), assuming a particular integral of the form $c \cos\mu t + c' \sin\mu t$ and determining the constants c, c' by substituting this expression in (12). Discuss the result.

3. Examples of curvilinear motion.

298. Central Forces. The motion of a particle in the field of a central force has been studied in Kinematics, under central motion, Arts. 96–113. It will here suffice to add certain further developments that are best expressed in dynamical terms.

299. Force Proportional to the Distance: $f(r) = \kappa^2 r$. The equations of motion (2) are in this case

$$\ddot{x} = \mp \kappa^2 x, \quad \ddot{y} = \mp \kappa^2 y,$$

the upper sign holding for attraction, the lower for repulsion. Their solution is very simple, because each equation can be integrated separately. We find, in the case of *attraction*,

$$x = a_1 \cos\kappa t + a_2 \sin\kappa t, \quad y = b_1 \cos\kappa t + b_2 \sin\kappa t,$$

and in the case of *repulsion*,

$$x = a_1 e^{\kappa t} + a_2 e^{-\kappa t}, \quad y = b_1 e^{\kappa t} + b_2 e^{-\kappa t};$$

a_1, a_2, b_1, b_2, being the constants of integration.

To find the equation of the orbit, it is only necessary to eliminate t in each case.

In the case of attraction, this elimination can be performed by solving for $\cos\kappa t$, $\sin\kappa t$, squaring and adding. The result is

$$(a_1 y - b_1 x)^2 + (a_2 y - b_2 x)^2 = (a_1 b_2 - a_2 b_1)^2,$$

and this represents an ellipse, since

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2 \equiv (a_1 b_2 - a_2 b_1)^2$$

is always positive. The center of the ellipse is at the origin, and the lines $a_1 y = b_1 x$, $a_2 y = b_2 x$ are a pair of conjugate diameters.

In the case of repulsion, solve for $e^{\kappa t}$ and $e^{-\kappa t}$, and multiply. The resulting equation,

$$(a_1 y - b_1 x)(b_2 x - a_2 y) = (a_1 b_2 - a_2 b_1)^2,$$

represents a hyperbola whose asymptotes are the lines $a_1 y = b_1 x$, $a_2 y = b_2 x$.

300. It is worthy of notice that the more general problem of *the motion of a particle attracted by any number of fixed centers, with forces directly proportional to the distances from these centers,* can be reduced to the problem of Art. 299.

Let x, y, z be the co-ordinates of the particle, r_i its distance from the center O_i; x_i, y_i, z_i the co-ordinates of O_i; and $\kappa_i^2 r_i$ the acceleration produced by O_i. Then the x-component of the resultant acceleration is

$$= -\Sigma\kappa_i^2 r_i \cdot \frac{x - x_i}{r_i} = -\Sigma\kappa_i^2(x - x_i) = -x\Sigma\kappa_i^2 + \Sigma\kappa_i^2 x_i;$$

and similar expressions obtain for the y and z components. Hence, the equations of motion are

$$\ddot{x} = -x\Sigma\kappa_i^2 + \Sigma\kappa_i^2 x_i, \quad \ddot{y} = -y\Sigma\kappa_i^2 + \Sigma\kappa_i^2 y_i, \quad \ddot{z} = -z\Sigma\kappa_i^2 + \Sigma\kappa_i^2 z_i.$$

As the right-hand members are linear in x, y, z, there is one, and only one, point at which the resultant acceleration is zero. Denoting its co-ordinates by $\bar{x}$, $\bar{y}$, $\bar{z}$, we have

$$\bar{x} = \frac{\Sigma\kappa_i^2 x_i}{\Sigma\kappa_i^2}, \quad \bar{y} = \frac{\Sigma\kappa_i^2 y_i}{\Sigma\kappa_i^2}, \quad \bar{z} = \frac{\Sigma\kappa_i^2 z_i}{\Sigma\kappa_i^2}.$$

The form of these equations shows that this point of zero acceleration which is sometimes called the *mean center* is the centroid of the centers of force, if these centers be regarded as containing masses equal to κ_i^2. It is evidently a fixed point.

By introducing the co-ordinates of the mean center, we can reduce the equations of motion to the simple form

$$\ddot{x} = -\kappa^2(x - \bar{x}), \quad \ddot{y} = -\kappa^2(y - \bar{y}), \quad \ddot{z} = -\kappa^2(z - \bar{z}),$$

where $\kappa^2 = \Sigma\kappa_i^2$. Finally, taking the mean center as origin, we have

$$\ddot{x} = -\kappa^2 x, \quad \ddot{y} = -\kappa^2 y, \quad \ddot{z} = -\kappa^2 z.$$

It thus appears that *the motion of the particle is the same as if there were only a single center of force*, viz., the mean center (x, y, z), *attracting with a force proportional to the distance from this center.*

The plane of the orbit is, of course, determined by the mean center and the initial velocity.

301. It is easy to see that most of the considerations of Art. 300 apply even when some or all of the centers *repel* the particle with forces proportional to the distance. It may, however, happen in this case that the mean center lies at infinity, in which case, of course, it can not be taken as origin.

Simple geometrical considerations can also be used to solve such problems. Thus, in the case of two attractive centers O_1, O_2 (Fig. 73) of equal intensity κ^2, the forces can evidently be represented by the distances $PO_1 = r_1$, $PO_2 = r_2$ of the particle P from the centers. Their resultant is therefore $= 2PO$, if O denotes the point midway between O_1 and O_2; and this resultant always passes through this fixed point O, and is proportional to the distance PO from this point.

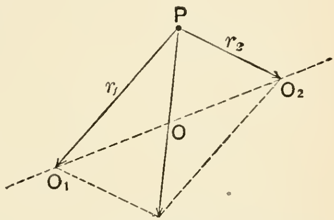

Fig. 73.

302. Exercises.

(1) Determine the constants of integration in Art. 299, if x_0, y_0 are the co-ordinates of the particle at the time $t = 0$ and v_1, v_2 the components of its velocity v_0 at the same time. The equation of the orbit will assume the form

$$\kappa^2(x_0 y - y_0 x)^2 + (v_1 y - v_2 x)^2 = (x_0 v_2 - y_0 v_1)^2$$

for attraction, and

$$\kappa^2(x_0 y - y_0 x)^2 - (v_1 y - v_2 x)^2 = -(x_0 v_2 - y_0 v_1)^2$$

for repulsion.

(2) Show that the semi-diameter conjugate to the initial radius vector has the length v_0/κ, where $v_0^2 = v_1^2 + v_2^2$. As any point of the orbit can be regarded as initial point, it follows that *the velocity at any point is proportional to the parallel diameter of the orbit.*

(3) Find what the initial velocity must be to make the orbit a circle in the case of attraction, and an equilateral hyperbola in the case of repulsion.

(4) The initial radius vector r_0 and the initial velocity v_0 being given geometrically, show how to construct the axes of the orbit described

under the action of a central force (of given intensity κ^2) proportional to the distance from the origin.

(5) A particle describes an ellipse under the action of a central force proportional to the distance; show that the eccentric angle is proportional to the time, and find the corresponding relation for a hyperbolic orbit.

(6) A particle of mass m describes a conic under the action of a central force $F = \mp m\kappa^2 r$. Show that the sectorial velocity is $\frac{1}{2}c = \frac{1}{2}\kappa ab$, a and b being the semi-axes of the conic.

(7) In Ex. (6) show that the time of revolution is $T = 2\pi/\kappa$, if the conic is an ellipse.

(8) A particle describes a conic under the action of a force whose direction passes through the center of the conic. Show that the force is proportional to the distance from the center.

(9) A particle is acted upon by two central forces of the same intensity (κ^2), each proportional to the distance from a fixed center. Determine the orbit: (a) when both forces are attractive; (b) when both are repulsive; (c) when one is an attraction, the other a repulsion.

(10) A particle of mass m is attracted by two centers O_1, O_2 of equal mass m' and repelled by a third center O_3, whose mass is $m'' = 2m'$. If the forces are all directly proportional to the respective distances, determine and construct the orbit.

(11) When a particle moves in an ellipse under a force directed towards the center, find the time of moving from the end of the major axis to a point whose polar angle is θ.

(12) Prove that if, in the problem of Art. 301, the intensities of O_1 and O_2 are κ_1, κ_2, the resultant attraction F passes through the centroid G of two masses κ_1 κ_2, placed at O_1, O_2, and that $F = (\kappa_1 + \kappa_2)PG$.

(13) In Art. 299, in the case of attraction, the component motions are evidently simple harmonic oscillations. Show that the equation of the path can be put in the form (comp. Art. 89)

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\sin\delta + \frac{y^2}{b^2} = \cos^2\delta.$$

(14) Show that the total energy of a particle of mass m describing an ellipse of semi-axes a, b under a force $m\kappa^2 r$ directed to the center is $= \frac{1}{2}m\kappa^2(a^2 + b^2)$.

303. Force Inversely Proportional to the Square of the Distance: $f(r) = \mu/r^2$ (Newton's law).

It has been shown in Kinematics (Arts. 99–108) how this law of acceleration can be deduced from Kepler's laws of planetary motion. From Kepler's first law Newton concluded that the acceleration of a planet (regarded as a point of mass m) is constantly directed towards the sun; from the second he found that this acceleration is inversely proportional to the square of the distance. The motion of a planet can therefore be explained on the hypothesis of an attractive force,

$$F = m\frac{\mu}{r^2},$$

issuing from the sun

The value of μ, which represents the acceleration at unit distance or the so-called intensity of the force, was found to be (Art. 108; or below, Art. 315)

$$\mu = 4\pi^2\frac{a^3}{T^2};$$

and as, according to Kepler's third law, the quantity a^3/T^2 has the same value for all the planets, Newton inferred that the intensity of the attracting force is the same for all planets; in other words, that it is one and the same central force that keeps the different planets in their orbits.

304. It was further shown by Newton and Halley that the motions of the comets are due to the same attractive force. The orbits of the comets are generally ellipses of great eccentricity, with the sun at one of the foci. As a comet is within range of observation only while in that portion of its path which lies nearest to the sun, a portion of a parabola, with the same focus and vertex, can be substituted for this portion of the elliptic orbit, as a first approximation.

It is also found from observation that the motions of the moons or satellites around the planets follow very nearly Kepler's laws. A planet can therefore be regarded as attracting each of its satellites with a force proportional to the mass of the satellite and inversely proportional to the square of the distance.

305. All these facts led Newton to suspect that the force of terrestrial gravitation, as observed in the case of falling bodies on the earth's surface, might be the same as the force that keeps the moon in its orbit around the earth. This inference could easily be tested, since the acceleration g of falling bodies as well as the moon's distance and time of revolution were known.

Let m be the mass of the moon, a the major semi-axis of its orbit, T the time of revolution, r the distance between the centers of earth and moon; then the earth's attraction on the moon is (Art. 303)

$$F = 4\pi^2 m \frac{a^3}{T^2 r^2},$$

or, since the eccentricity of the moon's orbit is so small that the orbit can be regarded as nearly circular, $F = 4\pi^2 ma/T^2$. On the other hand, the attraction exerted by the earth on a mass m on its surface, *i. e.* at the distance $R = 3963$ miles from the center, is $F' = mg$. Now, if these forces are actually in the inverse ratio of the squares of the distances, we must have

$$\frac{F'}{F} = \frac{a^2}{R^2},$$

or, since the distance of the moon is nearly $= 60R$, $F' = 60^2 F$. Substituting the above values of F and F', we find

$$g = 4\pi^2 \cdot \frac{60^3 R}{T^2}.$$

With $R = 3963$ miles, $T = 27^d\ 7^h\ 43^m$, this gives $g = 32.0$, a value which agrees sufficiently with the observed value of g, considering the rough degree of approximation used.

306. In this way Newton was finally led to his **law of universal gravitation,** which asserts that *every particle of mass m attracts every other particle of mass m′ with a force*

$$F = \kappa \frac{mm'}{r^2},$$

where r is the distance of the particles and κ a constant, viz. the acceleration produced by a unit of mass in a unit of mass at unit distance (see Arts. 245, 246).

The best test of this hypothesis as an actual law of physical nature is found in the close agreement of the results of theoretical astronomy based on this law with the observed celestial phenomena.

307. Taking Newton's law as a basis, let us now turn to the converse problem of *determining the motion of a particle acted upon by a single central force for which* $f(r) = \mu/r^2$ (problem of planetary motion).

It has been shown in Kinematics (Arts. 109–112) that *if the force be attractive*, the particle will describe a conic section with one of the foci at the center of force, the conic being an ellipse, parabola, or hyperbola, according as

$$v_0^2 \lesseqqgtr \frac{2\mu}{r_0}. \tag{13}$$

If the force be repulsive, the same reasoning will apply, except that μ is then a negative quantity. The orbit is, therefore, in this case always hyperbolic; the branch of the hyperbola that forms the orbit must evidently turn its convex side towards the focus at which the center of force is situated, since the force always lies on the concave side of the path.

308. To exhibit fully the determination of the constants and the dependence of the nature of the orbit on the initial conditions, a solution somewhat different from that given in Kinematics will here be given for the problem of planetary motion in its simplest form.

With $f(r) = \mu/r^2$, the equation of kinetic energy and work (5) Art. 271, gives (comp. (19), Art. 109)

$$v^2 = v_0^2 - 2\mu \int_{r_0}^{r} \frac{dr}{r^2} \quad v_0^2 + \frac{2\mu}{r} - \frac{2\mu}{r_0},$$

or, if the constant of integration be denoted briefly by h and $u = 1/r$ be introduced,

$$v^2 = 2\mu u + h, \text{ where } h = v_0^2 - \frac{2\mu}{r_0}. \tag{14}$$

Substituting this expression for v^2 in the equation (15), Art. 105, we find the differential equation of the orbit in the form

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{c^2}(2\mu u + h), \tag{15}$$

or

$$\left(\frac{du}{d\theta}\right)^2 = -\left(u - \frac{\mu}{c^2}\right)^2 + \frac{\mu^2}{c^4} + \frac{h}{c^2}.$$

To integrate, we introduce a new variable u' by putting

$$u - \frac{\mu}{c^2} = u' \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}};$$

the resulting equation,

$$\left(\frac{du'}{d\theta}\right)^2 = 1 - u'^2, \text{ or } d\theta = \pm \frac{du'}{\sqrt{1 - u'^2}},$$

has the general integral

$$\theta - \alpha = \mp \cos^{-1} u', \text{ or } u' = \cos(\theta - \alpha),$$

where α is the constant of integration. The orbit has, therefore, the equation

$$\frac{1}{r} = \frac{\mu}{c^2} + \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}} \cos(\theta - \alpha), \tag{16}$$

which agrees with the equation (24) given in Kinematics, Art. 112, excepting the different notation used for the constants.

309. The equation (16) represents a conic section referred to its focus as origin. The general focal equation of a conic is

$$\frac{1}{r} = \frac{1}{l} + \frac{e}{l} \cos(\theta - \alpha), \tag{17}$$

where l is the semi-latus rectum, or parameter, e the eccentricity, and α the angle made with the polar axis by the line joining the focus to the nearest vertex.

In a planetary orbit (Fig. 74), the sun S being at one of the foci, the nearest vertex A is called the *perihelion*, the other vertex A' the *aphelion*, and the angle $\theta - \alpha$ made by any radius vector $SP = r$ with the perihelion distance SA is called the *true anomaly*.

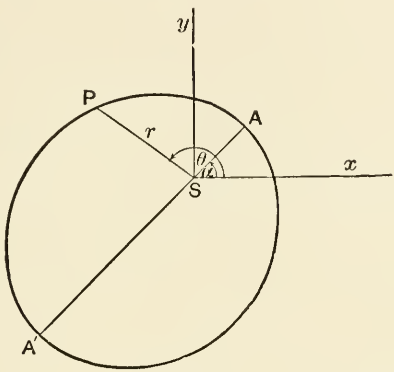

Fig. 74.

Comparing equations (17) and (16), we find, for the determination of the constants:

$$\frac{1}{l} = \frac{\mu}{c^2}, \quad \frac{e}{l} = \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}};$$

hence,

$$l = \frac{c^2}{\mu}, \quad e = \sqrt{1 + \frac{hc^2}{\mu^2}}, \tag{18}$$

or, solving for c and h,

$$c = \sqrt{\mu l}, \quad h = \mu \frac{e^2 - 1}{l}. \tag{19}$$

310. The expression for the eccentricity e in (18) determines the nature of the conic; the orbit is an ellipse, parabola, or hyperbola, according as $e \lesseqgtr 1$; hence, by (18), according as the constant h of the equation of kinetic energy is negative, zero, or positive. Owing to the value of h given in (14), this criterion agrees with the form (13), Art. 307.

It should be observed that it follows from (13) that *the nature* of the conic is independent of the *direction* of the initial velocity.

The criterion (13) can be given the following interpretation. Con-

sider a particle attracted by a fixed center according to Newton's law. If it move in a straight line passing through the center, the principle of kinetic energy gives for its velocity, at the distance r,

$$v^2 = v_0^2 - 2\mu \int_{r_0}^{r} \frac{dr}{r^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0};$$

hence, if it start from rest at an infinite distance from the center, it would acquire the velocity $\sqrt{2\mu/r}$ at the distance r. The criterion (13) is therefore equivalent to saying that *the orbit is an ellipse, a parabola, or a hyperbola, according as the velocity at any point is less than, equal to, or greater than the velocity which the particle would have acquired at that point by falling towards the center from infinity.*

311. For a *central conic*, whose axes are $2a$, $2b$, we have $l = b^2/a$. $e = \sqrt{a^2 \mp b^2}/a$ (the upper sign relating to the ellipse, the lower to the hyperbola), so that the equations (19) reduce to the following:

$$c = b\sqrt{\frac{\mu}{a}}, \quad h = \mp\frac{\mu}{a}. \tag{20}$$

The latter relation, with the value of h from (14), gives for the major or focal semi-axis a:

$$\pm\frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}; \tag{21}$$

while the former, with the value of c as given in Art. 100, determines the minor or transverse axis b:

$$b = c\sqrt{\frac{a}{\mu}} = r_0 v_0 \sin\psi_0 \sqrt{\frac{a}{\mu}}. \tag{22}$$

312. The magnitudes of the axes having thus been found, their directions can be determined by a simple construction which furnishes the second focus.

In the *ellipse*, the focal radii have a constant sum $= 2a$, and lie on the same side of the tangent, making equal angles with it. In the *hyperbola*, they have a constant difference $= 2a$, and lie on opposite sides of the tangent.

Hence, determining the point O'' (Fig. 75), which is symmetrical to the center of force O with respect to the initial velocity, and drawing the line P_0O'', we have only to lay off on this line from P_0 a length $P_0O' = \pm(2a - r_0)$; then O' is the second focus, which for an elliptic

orbit must be taken with O on the same side of the tangent P_0T, and for a hyperbolic orbit on the opposite side.

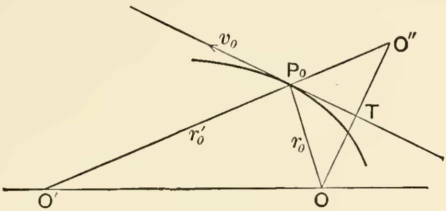

Fig. 75.

313. For a *parabola*, since $e = 1$, we find, from (19),

$$h = 0, \; l = \frac{c^2}{\mu} = \frac{v_0^2 r_0^2 \sin^2\psi_0}{\mu}. \tag{23}$$

The axis of the parabola is readily found by remembering that the perpendicular let fall from the focus on the tangent bisects the tangent (*i. e.* the segment of the tangent between the point of contact and the axis). Hence, if OT (Fig. 76) be the perpendicular let fall from the center O on the velocity v_0, it is only necessary to make $TT' = P_0T$, and T' will be a point of the axis. Moreover, the perpendicular let fall from T on OT' will meet the axis at the vertex A of the parabola, so that $OA = \frac{1}{2}l$.

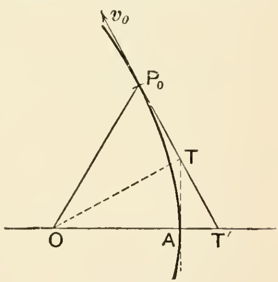

Fig. 76.

314. The relation (21), which must evidently hold at *any* point of the orbit, can be written in the form

$$v^2 = 2\mu\left(\frac{1}{r} \mp \frac{1}{2a}\right), \tag{24}$$

the upper sign relating to the ellipse, the lower to the hyperbola, while for the parabola, the second term in the parenthesis vanishes (since $a = \infty$).

This convenient expression for the velocity in terms of the radius vector might have been derived directly from the fundamental relation (Art. 100) $v = c/p$, the first of the equations (19), $c^2 = \mu l$, and the

geometrical properties of the conic sections ($r \pm r' = 2a$, $pp' = b^2$, $p'r = pr'$, where r, r' are the focal radii, and p, p' the perpendiculars let fall from the foci on the tangent). The proof is left to the student.

315. Time. In the case of an elliptic orbit, the time T of a complete revolution, usually called the **periodic time**, is found by remembering that the sectorial velocity is constant and $= \frac{1}{2}c$, whence

$$T = \frac{2\pi ab}{c},$$

or, by (20),

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{n}. \tag{25}$$

The constant

$$n = \sqrt{\frac{\mu}{a^3}},$$

which evidently represents the mean angular velocity about the center in one revolution, is called the **mean motion** of the planet. It should be noticed that it depends not only on the intensity of the force, but also on the major axis of the orbit, while in the case of a force directly proportional to the distance the periodic time is independent of the size of the orbit (see Art. 302, Ex. 7).

The periodic time T and the major axis a of a planetary orbit determine the intensity μ of the force:

$$\mu = 4\pi^2 \frac{a^3}{T^2}, \tag{26}$$

whence

$$F = mf(r) = m\frac{\mu}{r^2} = 4\pi^2 m \frac{a^3}{T^2 r^2}, \tag{27}$$

where m is the mass of the planet.

316. To find generally the time t in terms of θ or r, it is best to introduce the eccentric angle ϕ of the ellipse as a new variable, and to express t, r, and θ in terms of ϕ. In astronomy, the polar angle θ is known as the *true anomaly*, and the eccentric angle ϕ as the *eccentric anomaly*.

The relation of the eccentric angle ϕ to the polar co-ordinates r, θ will appear from Fig. 77, in which P is the position of the planet at the time t, P' the corresponding point on the circumscribed circle, $\measuredangle AOP = \theta$ the true anomaly, and $\measuredangle ACP' = \phi$ the eccentric anomaly. The focal equation of the ellipse

$$r = \frac{l}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

gives $r + er\cos\theta = a - ae^2$; and the figure shows that $r\cos\theta = a\cos\phi - ae$; hence

$$r = a(1 - e\cos\phi), \quad \text{or} \quad a - r = ae\cos\phi. \tag{28}$$

Equating this value of r to that given by the polar equation of the ellipse, we have

$$1 - e\cos\phi = \frac{1 - e^2}{1 + e\cos\theta}, \quad \text{or} \quad \cos\theta = \frac{\cos\phi - e}{1 - e\cos\phi}.$$

A more symmetrical form can be given to this relation by computing

$$1 - \cos\theta \equiv 2\sin^2\tfrac{1}{2}\theta = (1 + e)\frac{1 - \cos\phi}{1 - e\cos\phi},$$

$$1 + \cos\theta \equiv 2\cos^2\tfrac{1}{2}\theta = (1 - e)\frac{1 + \cos\phi}{1 - e\cos\phi};$$

whence, by division,

$$\tan\tfrac{1}{2}\theta = \sqrt{\frac{1 + e}{1 - e}}\tan\tfrac{1}{2}\phi. \tag{29}$$

317. To find t in terms of r, we have only to substitute in (24) for v^2 its value (Art. 105), and to integrate the resulting differential equation

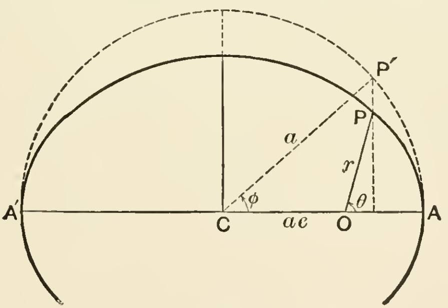

Fig. 77.

$$\left(\frac{dr}{dt}\right)^2 + \frac{c^2}{r^2} = \frac{2\mu}{r} - \frac{\mu}{a}.$$

As, by (20), Art. 311, $c^2 = \mu b^2/a = \mu a(1 - e^2)$, this equation becomes

$$r^2\left(\frac{dr}{dt}\right)^2 = \frac{\mu}{a}[a^2e^2 - (a - r)^2],$$

or

$$dt = \sqrt{\frac{a}{\mu}}\frac{rdr}{\sqrt{a^2e^2 - (a - r)^2}}.$$

The integration is easily performed by introducing the eccentric angle ϕ as variable by means of (28); this gives

$$dt = \sqrt{\frac{a}{\mu}} \cdot a(1 - e \cos\phi)d\phi.$$

If the time be counted from the perihelion passage of the planet, we have $t = 0$ when $r = a - ae$, *i. e.* when $\phi = 0$; hence, putting $\sqrt{\mu/a^3} = n$, as in Art. 315, we find

$$nt = \phi - e \sin\phi. \tag{30}$$

This relation is known as *Kepler's equation;* the quantity nt is called the *mean anomaly.*

318. Kepler's equation (30) can be derived directly by considering that the ellipse APA' (Fig. 77) can be regarded as the projection of the circle $AP'A'$, after turning this circle about AA' through an angle $= \cos^{-1}(b/a)$. For it follows that the elliptic sector AOP is to the circular sector AOP' as b is to a. Now, for the circular sector we have

$$AOP' = ACP' - OCP' = \tfrac{1}{2}a^2\phi - \tfrac{1}{2}ae \cdot a \sin\phi = \tfrac{1}{2}a^2(\phi - e \sin\phi);$$

hence, the elliptic sector described in the time t is

$$AOP = \frac{b}{a} \cdot AOP' = \tfrac{1}{2}ab\ (\phi - e\ \sin\phi).$$

The sectorial velocity being constant by Kepler's first law, we have

$$\frac{AOP}{t} = \frac{\pi ab}{T};$$

hence,

$$t = \frac{T}{2\pi}(\phi - e \sin\phi),$$

and this agrees with (30) since, by (25), $2\pi/T = n$.

319. Kepler's equation (30) gives the time as a function of ϕ; by means of (28), it establishes the relation between t and r; by means of (29), it connects t with θ. It is, however, a transcendental equation and cannot be solved for ϕ in a finite form.

For orbits with a small eccentricity e, an approximate solution can be obtained by writing the equation in the form

$$\phi = nt + e\ \sin\phi,$$

and substituting under the sine for ϕ its approximate value nt:

$$\phi = nt + e \sin nt. \tag{31}$$

This amounts to neglecting terms containing powers of e above the first power.

Substituting this value of ϕ in (28), we have with the same approximation

$$r = a(1 - e \cos nt). \tag{32}$$

To find θ in terms of t, we have from the equation of the ellipse, $r = a(1 - e^2)(1 + e \cos\theta)^{-1} = a(1 - e \cos\theta)$, neglecting again terms in e^2; hence, $r^2 = a^2(1 - 2e \cos\theta)$. Substituting this value in the equation of areas, $r^2 d\theta = c dt = \sqrt{\mu a(1 - e^2)}dt$, we find

$$(1 - 2e \cos\theta)d\theta = \sqrt{\frac{\mu}{a^3}}\, dt = n dt;$$

whence, by integration, since $\theta = 0$ for $t = 0$,

$$\theta - 2e \sin\theta = nt,$$

or finally,

$$\theta = nt + 2e \sin nt. \tag{33}$$

Thus we have in (31), (32), (33) approximate expressions for ϕ, r, and θ directly in terms of the time. The quantity $2e \sin nt$, by which the true anomaly θ exceeds the mean anomaly nt, is called the *equation of the center*.

320. Exercises.

(1) A particle is attracted by a fixed center according to Newton's law. What must be the initial velocity if the orbit is to be circular?

(2) A number of particles are projected, from the same point in the field of a force following Newton's law, with the same velocity, but in different directions. Show that the periodic times are the same for all the particles.

(3) The mean distance of Mars from the sun being 1.5237 times that of the earth, what is the time of revolution of Mars about the sun?

(4) A particle describes a conic under the action of a central force following Newton's law; if the intensity μ of the force be suddenly changed to μ', what is the effect on the orbit?

(5) In Ex. (4), if the original orbit was a parabola and the intensity be doubled, what is the new orbit?

(6) Regarding the moon's orbit about the earth as circular, what would it become: (*a*) if the earth's mass were suddenly doubled? (*b*) if it were reduced to one half?

(7) In Ex. (4), determine the effect on the major semi-axis (or "mean distance") a and on the periodic time T, of a *small* change in the intensity μ of the force.

(8) If the mass M of the sun be suddenly increased by M/n, n being very large, while the earth is at the end of the minor axis of its orbit, what would be the effect on the earth's mean distance and on the period of revolution T?

(9) Find the equation of the hodograph of planetary motion, derive from it the expression for the velocity in terms of the radius vector, and show that the velocity is a maximum in perihelion and a minimum in aphelion.

(10) Show that the greatest velocity of a planet in its orbit about the sun is to its least velocity as $1 + e$ is to $1 - e$; and find this ratio for the earth, whose orbit has the eccentricity $e = 0.016\,771\,2$.

(11) Find the time exactly as a function of θ, for a parabolic orbit.

(12) The latus rectum passing through the sun divides the earth's orbit into two different parts; in what time are these described if the whole time is 365¼ days?

(13) Show that the path of a projectile *in vacuo* is an ellipse, parabola, or hyperbola, according as $v_0 \lesseqgtr 36{,}800$ ft. per second (= 7 miles per second, nearly). One of the foci lies at the center of the earth, and the ordinary assumption that the path is parabolic means that this center can be regarded as infinitely distant. Show also that the path becomes circular for $v_0 = 5$ miles per second, nearly.

321. The Problem of Two Bodies. In the preceding discussion of the motion of a particle under the action of a central force, it has been assumed that the center of force is fixed. In the applications of the theory of central forces this assumption is in general not satisfied. Thus, in considering the motion of a planet around the sun, the force of attraction is, according to Newton's law of universal gravitation (Art. 306), regarded as due to the presence of a mass M at the center (sun), and of a mass m at the attracted point (planet); and the action between these two masses is a mutual action, being of the nature of a *stress*, *i. e.* consisting of two equal and opposite forces, each equal to

$$F = \kappa \frac{mM}{r^2}.$$

Hence, the mass m of the planet attracts the mass M of the sun with

precisely the same force with which the mass M of the sun attracts the mass m of the planet. The attraction affects, therefore, the motions of both bodies.

322. The *accelerations* produced by the two forces are, of course, not equal. Indeed, the acceleration $F/m = \kappa M/r^2$, produced in the planet by the sun, is very much greater than the acceleration $F/M = \kappa m/r^2$, produced by the planet in the sun; for the mass of even the largest planet (Jupiter) is less than one thousandth of that of the sun. The assumption of a fixed center can therefore be regarded as a first approximation in the problem of the motion of a planet about the sun.

In the case of the earth and moon, the difference of the masses is not so great, the mass of the moon being nearly one eightieth of that of the earth.

It can be shown, however, that the results deduced on the assumption of a fixed center can, by a simple modification, be made available for the solution of *the general problem of the motions of two particles of masses m, M, subject to no forces besides their mutual attraction.* In astronomy, this is called the **problem of two bodies.** In the solution below we assume the attraction to follow Newton's law of the inverse square of the distance. It will be convenient to speak of the two particles, or bodies, as planet (m) and sun (M).

323. With regard to any fixed system of rectangular axes, let x, y, z be the co-ordinates of the planet (m), at the time t; x', y', z' those of the sun (M), at the same time; so that for their distance r we have

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

Then the equations of motion of the planet are

$$m\ddot{x} = F \cdot \frac{x' - x}{r}, \quad m\ddot{y} = F \cdot \frac{y' - y}{r}, \quad m\ddot{z} = F \cdot \frac{z' - z}{r}, \tag{1}$$

while the equations of motion of the sun are

$$M\ddot{x}' = F \cdot \frac{x - x'}{r}, \quad M\ddot{y}' = F \cdot \frac{y - y'}{r}, \quad M\ddot{z}' = F \cdot \frac{z - z'}{r}. \tag{2}$$

By adding the corresponding equations of the two sets, we find

$$\frac{d^2}{dt^2}(mx + Mx') = 0, \quad \frac{d^2}{dt^2}(my + My') = 0, \quad \frac{d^2}{dt^2}(mz + Mz') = 0.$$

If it be remembered that the centroid of the two masses m, M has the co-ordinates

$$\bar{x} = \frac{mx + Mx'}{m + M}, \quad \bar{y} = \frac{my + My'}{m + M}, \quad \bar{z} = \frac{mz + Mz'}{m + M},$$

it appears that these equations can be written in the form

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \quad \frac{d^2\bar{z}}{dt^2} = 0;$$

in words: *the acceleration of the common centroid of planet and sun is zero; i. e. this centroid moves with constant velocity in a straight line.*

324. The integration of the equations (1) would give the absolute path of the planet. But the constants could not be determined, because the absolute initial position and velocity of the planet are, of course, not known. The same holds for the absolute path of the sun. All we can do is to determine the *relative* motion, and we proceed to find the motion of the planet relative to the sun.

Taking the sun's center as new origin for parallel axes, we have for the co-ordinates ξ, η, ζ of the planet in this new system,

$$\xi = x - x', \quad \eta = y - y', \quad \zeta = z - z'.$$

Now, dividing the equations (1) by m, the equations (2) by M, and subtracting the equations of set (2) from the corresponding equations of set (1), we find for the relative acceleration of the planet

$$\ddot{\xi} = -\kappa\frac{M + m}{r^2}\cdot\frac{\xi}{r}, \quad \ddot{\eta} = -\kappa\frac{M + m}{r^2}\cdot\frac{\eta}{r}, \quad \ddot{\zeta} = -\kappa\frac{M + m}{r^2}\cdot\frac{\zeta}{r}. \qquad (3)$$

The form of these equations shows that *the relative motion of the planet with respect to the sun is the same as if the sun were fixed and contained the mass $M + m$.* Thus the problem is reduced to that of a fixed center, the only modification being that the mass of the center M should be increased by that of the attracted particle m.

325. This result can also be obtained by the following simple consideration. The *relative* motion of the planet with respect to the sun would obviously not be altered if geometrically equal accelerations were applied to both. Let us, therefore, subject each body to an additional acceleration equal and opposite to the actual acceleration of the sun (whose components are obtained by dividing the equations (2) by M). Then the sun will be reduced to equilibrium, while the resulting acceleration of the planet, which is its relative acceleration with respect to the sun, will evidently be the sum of the acceleration exerted on it by

the sun and the acceleration exerted on the sun by the planet. This is just the result expressed by the equations (3).

326. It can here only be mentioned in passing that, while the problem of two bodies thus leads to equations that can easily be integrated, *the problem of three bodies* is one of exceeding difficulty, and has been solved only in a few very special cases. Much less has it been possible to integrate the $3n$ equations of the problem of n bodies.

327. According to the equations (3), the first and second laws of Kepler can be said to hold for the *relative* motion of a planet about the sun (or of a satellite about its primary). The third law of Kepler requires some modification, since the intensity of the center μ should not be κM, but $\kappa(M + m)$. We have, by (26), Art. 315,

$$\mu = \kappa(M + m) = 4\pi^2 \frac{a^3}{T^2};$$

in other words, the quotient a^3/T^2 is not independent of the mass m of the planet.

Thus, if m_1, m_2 be the masses of two planets, a_1, a_2 the major semi-axes of their orbits, and T_1, T_2 their periodic times, we have

$$\frac{a_1^3/T_1^2}{a_2^3/T_2^2} = \frac{M + m_1}{M + m_2} = \frac{1 + m_1/M}{1 + m_2/M}.$$

This quotient is approximately equal to 1 if M is very large in comparison with both m_1 and m_2; hence, for the orbits of the planets about the sun, Kepler's third law is very nearly true.

CHAPTER XIV.

CONSTRAINED MOTION OF A PARTICLE.

1. Introduction.

328. A free particle is said to have three degrees of freedom (Art. 231) since three co-ordinates are required to determine its position, and each of these co-ordinates can vary independently of the other two.

If the co-ordinates of a moving particle are subjected to one condition, say

$$\varphi(x, y, z) = 0, \tag{1}$$

the particle is said to have one constraint and only two degrees of freedom. It can then only move on the surface (1), and its position on this surface can be assigned by two co-ordinates (such as latitude and longitude on a sphere).

If the co-ordinates are subjected to two conditions, say

$$\varphi(x, y, z) = 0, \quad \psi(x, y, z) = 0, \tag{2}$$

the particle has two constraints and but one degree of freedom. It can only move along the curve of intersection of the two surfaces (2), and its position on this curve can be assigned by a single co-ordinate (such as the arc of the curve).

Three such conditions would in general prevent the particle entirely from moving.

The surface or curve to which a particle is constrained may vary its position or even its shape in the course of the motion. The equations (1) and (2) would then contain t as a fourth independent variable. We shall, however, in general assume that the surface or curve is fixed.

329. A particle constrained to a surface can be regarded as the limit of a small piece of matter confined between two very near impenetrable surfaces. The constraint to a curve can be imagined as due to a narrow tube having the shape of the curve, or by imagining the particle as a bead sliding along a wire.

In these cases the constraint is *complete*. But it is easy to imagine *incomplete*, *i. e.* partial or one-sided, constraints of various kinds. Thus the rails compel a train to follow a definite curve, but they do not prevent it from being lifted off the track; a stone attached to a cord and swung around by the hand is not completely constrained to the surface of a sphere, but only prevented from passing outside of the sphere.

While complete constraints are generally expressed by equations, one-sided constraints can be expressed by *inequalities*. Thus, for the stone, the condition is that its distance r from the hand cannot become greater than the length l of the cord: $r \leqq l$. As soon, however, as r becomes less than l, the constraining action ceases and the stone becomes free. For this reason it is in general sufficient to consider constraining *equations*; but the nature of the constraint, whether complete or partial, must be taken into account to determine when and where the constraint ceases to exist.

330. It is often convenient to replace the constraining conditions by introducing certain forces, called **reactions** of the constraining surface or curve (comp. Art. 232). Thus, in the case of the stone attached to the cord, we may imagine the cord cut and its tension introduced, to make the stone free.

If the constraints are thus replaced by the corresponding reactions, these unknown forces must be combined with the

given forces, and then the equations of motion of a free particle can be used. Thus, let X, Y, Z be the components of the resultant given force F, X', Y', Z' those of the resultant reaction F'; then the equations of motion are

$$m\ddot{x} = X + X', \quad m\ddot{y} = Y + Y', \quad m\ddot{z} = Z + Z'. \qquad (3)$$

In many applied problems the determination of these unknown reactions is more important than that of the actual motion. The term *Kinetostatics* has recently been proposed for this branch of mechanics.

2. Motion on a fixed curve.

331. Let us resolve the given force F and the constraining force F' each into a tangential component F_t, F_t' and a component F_n, F_n' in the normal plane. The normal component F_n' of the constraint is generally denoted by N and called the *normal reaction* of the curve; a force $-N$, equal and opposite to it, represents the normal *pressure* exerted by the particle on the curve. The tangential component F_t' of the constraint exists only if the curve is "rough," *i. e.* offers frictional resistance; denoting the coefficient of friction by μ we have (Art. 238) $F_t' = \mu N$.

Hence the equations of motion are:

$$m\dot{v} = F_t - \mu N, \quad m\frac{v^2}{\rho} = \text{res}\cdot(F_n, N). \qquad (4)$$

The former of these equations determines the actual motion along the given curve. The latter states that the forces F_n and N in the normal plane must have a resultant along the principal normal, toward the center of curvature, of magnitude mv^2/ρ; this resultant is called the *centripetal force*. A force $-mv^2/\rho$, equal and opposite to this resultant, is

called **centrifugal force**; it should be noticed that this is a force exerted not *on* the moving particle, but *by* it.

332. By the second of the equations (4), the centripetal force, mv^2/ρ, is the resultant of the given normal force F_n and the normal reaction N of the curve; see Fig. 78 whose

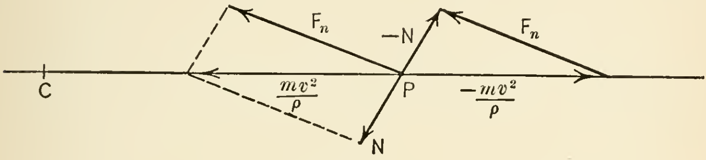

Fig. 78.

plane is the normal plane of the curve, P being the position of the particle and C the center of curvature.

It follows that *the pressure on the curve,* $-N$, *is the resultant of the given normal force* F_n *and the centrifugal force* $-mv^2/\rho$.

If in particular the given force F_n is zero, or at least negligible, as is often the case, the pressure on the curve is equal to the centrifugal force.

333. Denoting by N_x, N_y, N_z the components of the normal reaction N and observing that the frictional resistance μN is directed along the curve opposite to the sense of the motion we find that the equations (3) here assume the form

$$\begin{aligned} m\ddot{x} &= X + N_x - \mu N \frac{dx}{ds}, \\ m\ddot{y} &= Y + N_y - \mu N \frac{dy}{ds}, \qquad (5) \\ m\ddot{z} &= Z + N_z - \mu N \frac{dz}{ds}, \end{aligned}$$

where $N^2 = N_x^2 + N_y^2 + N_z^2$ and $N_x dx + N_y dy + N_z dz$

$= 0$ since N is normal to the path. In addition, we have of course the equations (2) of the curve.

Multiplying the equations (5) by dx, dy, dz and adding we find the equation of kinetic energy and work

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz - \mu N ds.$$

This relation might have been written down directly by considering that for a displacement ds along the fixed curve the normal reaction N does no work, while the work of friction is $-\mu N ds$.

If there be no friction ($\mu = 0$) it follows from the last equation, or from the first of the equations (4), that the velocity is independent of the reaction of the curve.

334. Exercises.

(1) A mass of 2 lbs. attached to a cord, 3 ft. long, is swung in a circle. Neglecting gravity, find the tension in pounds: (a) when the mass makes one revolution per second; (b) when it makes 8 revolutions per second. (c) If the cord cannot stand a tension of more than 300 lbs., what is the greatest allowable number of revolutions?

(2) A plummet is suspended from the roof of a railroad car; how much will it be deflected from the vertical when the train is running 45 miles an hour in a curve of 300 yards radius?

(3) A body on the surface of the earth partakes of the earth's daily rotation on its axis. The constraint holding it in its circular path is due to the attractive force of the earth. Taking the earth's equatorial radius as 3963 miles, show that the centripetal acceleration of a particle at the equator is about $\frac{1}{9}$ ft. per second, or about $\frac{1}{290}$ of the actually observed acceleration $g = 32.09$ of a body falling *in vacuo*.

(4) If the earth were at rest, what would be the acceleration of a body falling *in vacuo* at the equator?

(5) Show that if the velocity of the earth's rotation were over 17 times as large as it actually is, the force of gravity would not be sufficient to detain a body near the surface at the equator (comp. Ex. (13), Art. 320).

(6) Show that in latitude ϕ the acceleration of a falling body, if

the earth were at rest, would be $g_1 = g + j \cos^2\phi$, where g is the observed acceleration of a falling body on the rotating earth and j the centripetal acceleration at the equator. Thus, in latitude $\phi = 45°$, $g = 980.6$ cm.; hence $g_1 = 982.3$.

(7) Owing to the earth's rotation on its axis the direction of a plumb-line does not pass through the center of the earth, even when the earth, as here assumed, is regarded as a homogeneous sphere. Determine the angle δ of the deviation in latitude ϕ; in what latitude is δ greatest?

(8) A chandelier weighing 80 lbs. is suspended from the ceiling of a hall by means of a chain 12 ft. long whose weight is neglected. By how much is the tension of the chain increased if it be set swinging so that the velocity at the lowest point is 6 ft. per second?

335. *A particle of mass m subject to gravity alone is constrained to move in a vertical circle of radius l.* If there be no friction on the curve and the constraint be produced by a weightless rod or cord joining the particle to the center of the circle, we have the problem of the **simple mathematical pendulum.**

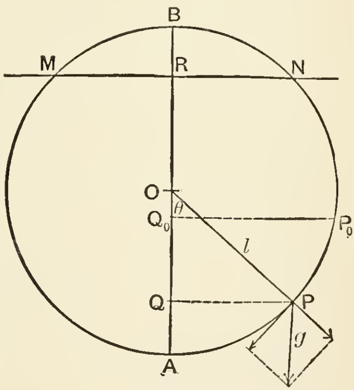

Fig. 79.

The first of the equations (4), Art. 331, is readily seen to reduce in this case (see Fig. 79) to the form

$$l \frac{d^2\theta}{dt^2} + g \sin\theta = 0.$$

A first integration gives, as shown in Kinematics (Arts. 63, 64),

$$\tfrac{1}{2}v^2 = g\left(l \cos\theta + \frac{v_0^2}{2g} - l \cos\theta_0\right),$$

where v_0 is the velocity which the particle has at the time $t = 0$ when its radius makes the angle $AOP_0 = \theta_0$ with the vertical. Multiplying by m, we have, for the kinetic energy of the particle,

$$\tfrac{1}{2}mv^2 = mg(l\cos\theta + h),$$

where $h = v_0^2/2g - l\cos\theta_0$ is a constant. If the horizontal line MN, drawn at the height $v_0^2/2g$ above the initial point P_0, intersect the vertical diameter AB at R, it appears from the figure that $h = RO$.

336. Taking R as origin and the axis of z vertically downwards, we have $RQ = z = l\cos\theta + h$; hence the force-function U has the simple expression

$$U = mgz;$$

and the velocity $v = \sqrt{2gz}$ is seen to become zero when the particle reaches the horizontal line MN.

For the further treatment of the problem, three cases must be distinguished according as this line of zero-velocity MN intersects the circle, touches it, or does not meet it at all; *i. e.* according as

$$h \lesseqqgtr l, \text{ or } \frac{v_0^2}{2g} \lesseqqgtr 2l\cos^2\tfrac{1}{2}\theta_0.$$

337. The second of the equations (4), Art. 331, serves to determine the reaction N of the circle, or the pressure $-N$ on the circle. We have

$$m\frac{v^2}{l} = -mg\cos\theta + N,$$

whence

$$N = m\left(\frac{v^2}{l} + g\cos\theta\right).$$

Substituting for v^2 its value from Art. 335, we find

$$N = mg\left(2\frac{h}{l} + 3\cos\theta\right).$$

The pressure on the curve has therefore its greatest value when $\theta = 0$, *i. e.* at the lowest point A. It becomes zero for $l\cos\theta_1 = -\tfrac{2}{3}h$, which is easily constructed.

338. If the constraint be complete as for a bead sliding along a circular wire, or a small ball moving within a tube, the pressure merely changes sign at the point $\theta = \theta_1$. But if the constraint be one-sided, the particle may at this point leave the circle. The one-sided constraint may be such that $OP \leqq l$, as when the particle runs in a groove cut on the inside of a ring, or when it is joined to the center by a cord; in this case the particle may leave the circle at some point of its upper half. Again, the one-sided constraint may be such that $OP \geqq l$, as when

the particle runs in a groove cut on the rim of a disk; in this case the particle can of course only move on the upper half of the circle.

339. Exercises.

(1) For $\theta_0 = 60°$, $l = 1$ ft., $v_0 = 9$ ft. per second, show that the particle will leave the circle very nearly at the point $\theta_1 = 120°$, if the constraint be such that $OP \leqq l$ (Art. 338).

(2) For $v_0 = 10$ ft. per second, everything else being as in Ex. (1) show that the particle will leave the circle at the point $\theta_1 = 134\frac{1}{2}°$, nearly.

(3) A particle, subject to gravity and constrained to the inside of a vertical circle ($OP \leqq l$), makes complete revolutions. Show that it cannot leave the circle at any point, if $\frac{2}{3}h > l$; and that it will leave the circle at the point for which $\cos\theta = -\frac{2}{3}h/l$, if $\frac{2}{3}h < l$.

(4) A particle subject to gravity moves on the outside of a vertical circle; determine where it will leave the circle: (*a*) if MN (Fig. 79) intersects the circle; (*b*) if MN touches the circle; (*c*) if MN does not meet the circle.

(5) A particle subject to gravity is compelled to move on any vertical curve $z = f(x)$ without friction. Show that the velocity at any point is $v = \sqrt{2gz}$ (comp. Art. 336) if the horizontal axis of x be taken at a height above the initial point equal to the "height due to the initial velocity," *i. e.* $v_0^2/2g$.

(6) A particle slides on the outside of a smooth vertical circle, starting from rest at the highest point of the circle. Find where it will meet the horizontal plane through the lowest point of the circle.

340. If for a particle constrained to a curve, under given forces, the time of reaching any particular point O is the same from whatever point of the curve the particle starts with zero velocity, the curve is called a **tautochrone** for the given forces, and the point O is called the *point of tautochronism.*

In a vertical plane, if gravity is the only force, a cycloid with vertical axis can be shown to be a tautochrone, with the vertex as point of tautochronism. This will even be true if the curve be rough, or if the particle be subject to a

resistance proportional to the velocity in the direction of motion; but, for the sake of simplicity, we exclude these complications.

The problem of determining a tautochrone for given forces (if such a curve exists) is rather different in nature from the ordinary problems of mechanics inasmuch as it is here required to find a curve, on which motions of a certain kind may take place. Indeed, it is a generalization of the problem of the tautochrone that led Abel to the first solution of an integral equation.*

341. With respect to a horizontal axis Ox and a vertical axis Oz through the point of tautochronism, the principle of kinetic energy and work (comp. Art. 339, Ex. 5) gives for the velocity

$$v^2 = 2g(h - z),$$

where h is the ordinate of the starting point P. Counting the arc s from O we have $ds/dt = -\sqrt{2g(h - z)}$, whence the time of motion from P to O:

$$t = -\int_{z=h}^{0} \frac{ds}{\sqrt{2g(h - z)}} = \frac{1}{\sqrt{2g}} \int_0^h \frac{ds}{\sqrt{h - z}}.$$

If we put $s = f(z)$ and hence $ds = f'(z)dz$, the problem requires the determination of the function $f(z)$ for which the integral has a value independent of h. To make the limits independent of h let us put $z = hy$; we then find

$$t = \frac{1}{\sqrt{2g}} \int_0^1 \frac{f'(hy)hdy}{\sqrt{h(1 - y)}} = \frac{1}{\sqrt{2g}} \int_0^1 f'(hy) \sqrt{\frac{hy}{(1 - y)y}}\, dy.$$

This integral will be independent of h if $f'(hy)\sqrt{hy}$ is

*See M. Bôcher, Integral equations, Cambridge, University Press, 1909, p. 6.

independent of h; and as this expression is symmetric in h and y, it will then be also independent of z. We can therefore put

$$f'(hy)\sqrt{hy} = f'(z)\sqrt{z} = \sqrt{\kappa},$$

whence

$$f'(z) = \sqrt{\frac{z}{\kappa}} = \frac{ds}{dz} = \sqrt{1 + \left(\frac{dx}{dz}\right)^2};$$

solving for dx we find (comp. Art. 20):

$$x = \int_0^z \sqrt{\frac{\kappa - z}{z}}\, dz = \sqrt{z(\kappa - z)} + \kappa \sin^{-1}\sqrt{\frac{z}{\kappa}}.$$

This is the equation of a cycloid with O as vertex and Oz as axis. Putting $z = \kappa \sin^2\frac{1}{2}\theta$, we find the equations of the cycloid in the form

$$x = \tfrac{1}{2}\kappa(\theta + \sin\theta), \quad z = \tfrac{1}{2}\kappa(1 - \cos\theta),$$

so that κ is the diameter of the generating circle.

For the time we find:

$$t = \sqrt{\frac{\kappa}{2g}}\int_0^1 \frac{dy}{\sqrt{y - y^2}} = \pi\sqrt{\frac{\kappa}{2g}}.$$

342. Exercises.

(1) For a heavy particle moving without friction on a cycloid with vertical axis, $x = a(\theta + \sin\theta)$, $z = a(1 - \cos\theta)$, show that the equation of motion is $\ddot{s} = -gs/4a$, s being the arc counted from the vertex. Hence, if $v = 0$ for $s = s_0$, $s = s_0 \cos\sqrt{g/4a}\, t$, which shows that the time of reaching the lowest point is independent of s_0.

(2) The involute of a cycloid being an equal cycloid, with its vertex at the cusp, its cusp on the axis, of the original cycloid, the particle in Ex. (1) can be constrained to the cycloid by means of a cord of length $2a$, attached to the cusp of the involute, and wrapping itself on a cylinder erected on the involute as base (*cycloidal pendulum*). Show that, if the particle starts from rest at the cusp of the original cycloid, the tension of the cord is twice the normal component of the weight of the particle.

(3) Prove that it is not possible to construct a tautochrone (for gravity) from P to O with O as point of tautochronism unless the slope of OA is in absolute value $\gtreqless 2/\pi$.

343. The cycloid (with vertical axis) has another remarkable property; it is the **brachistochrone**, or curve of quickest descent, for a particle subject to gravity. More definitely: two points P_1, P_2 being given we may inquire to what curve in their vertical plane must a heavy particle be constrained to reach in the shortest time the lower point P_2 if it starts from P_1 with a given velocity.

As the time is given by a definite integral the problem requires the determination of that curve $z = f(x)$ for which this integral becomes a minimum. This problem has given rise to the invention of the calculus of variations.

As the problem can hardly be solved satisfactorily without using the methods of this calculus we merely state that the required curve is the cycloid through the two points, without cusp between them and with vertical axis.*

3. Motion on a fixed surface.

344. The equations of motion of a particle constrained to a surface do not differ in form from the equations (5), Art. 333, for a particle constrained to a curve. The normal reaction

$$N = \sqrt{N_x^2 + N_y^2 + N_z^2}$$

being normal to the given surface $\varphi(x, y, z) = 0$, we have

$$\frac{N_x}{\dfrac{\partial\varphi}{\partial x}} = \frac{N_y}{\dfrac{\partial\varphi}{\partial y}} = \frac{N_z}{\dfrac{\partial\varphi}{\partial z}}.$$

A comparatively simple problem is that of the *conical* or *spherical pendulum*, *i. e.* of a particle subject to gravity and constrained to the surface of a sphere. But even this problem can not be treated without introducing elliptic integrals.

* See O. Bolza, Variationsrechnung, Leipzig, Teubner, 1909, p. 207.

4. The method of indeterminate multipliers.

345. The following brief discussion of the equations of motion of a constrained particle is not so much intended to furnish methods for solving particular problems, but rather as a preparation for, and an introduction to, the general methods of mechanics of systems of particles subject to conditions.

For this reason we shall here assume the absence of friction on the constraining surface or curve; but, on the other hand, it is desirable to generalize by assuming that the constraints are variable, that is, that the conditional equations (1) and (2), Art. 328, contain the time t explicitly.

346. D'Alembert's Principle. The ordinary equations of motion of a *free* particle,

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z, \tag{6}$$

where X, Y, Z are the components of the resultant R of the given forces, merely express the equality of this force R, as a vector, to the mass-acceleration mj, which is sometimes called the *effective force*. It follows that *if the reversed effective force* $-mj$, or its components $-m\ddot{x}$, $-m\ddot{y}$, $-m\ddot{z}$, *be combined with the given forces we have a system in equilibrium at the given instant.* This is the fundamental idea of d'Alembert's principle, as it is now generally used.

Owing to this idea we can apply to kinetic problems the statical conditions of equilibrium. Thus, in the case of the free particle, the conditions of equilibrium of the forces $X, Y, Z, -m\ddot{x}, -m\ddot{y}, -m\ddot{z}$ are

$$X - m\ddot{x} = 0, \quad Y - m\ddot{y} = 0, \quad Z - m\ddot{z} = 0,$$

and thus the equations of motion are found.

But the conditions of equilibrium can also be expressed

by means of the principle of virtual work. By Art. 266, the necessary and sufficient condition of equilibrium of the particle under the forces $-m\ddot{x}$, $-m\ddot{y}$, $-m\ddot{z}$, X, Y, Z is that

$$(-m\ddot{x} + X)\delta x + (-m\ddot{y} + Y)\delta y + (-m\ddot{z} + Z)\delta z = 0 \quad (7)$$

for any virtual displacement $\delta s(\delta x, \delta y, \delta z)$. Owing to the independence of δx, δy, δz, their coefficients must vanish separately, and we find again the equations (6). In other words, the single equation (7) is equivalent to the three equations (6).

347. One constraint. If the particle is subject to the condition or constraint

$$\varphi(x, y, z, t) = 0, \quad (8)$$

it must throughout its motion remain on the surface represented by this equation. To apply d'Alembert's principle let the particle be subjected to a virtual displacement δs. If this displacement be selected along the position of the surface at the time t, the work of the reaction (which is normal to the surface (8), and hence to δs, since we assume that there is no friction) will be zero. Hence the equation of motion is the same as for a free particle, viz. (7). But the displacement δs must be along the surface (8), or as we shall say, *compatible with the constraint*. This requires that δx, δy, δz *be selected so as to satisfy the relation*

$$\varphi_x\delta x + \varphi_y\delta y + \varphi_z\delta z = 0, \quad (9)$$

where the partial derivatives φ_x, φ_y, φ_z of φ with respect to x, y, z are calculated regarding t as constant since we want a displacement along the position of the surface (8) at the time t.

The equations (7) and (9) constitute the equations of motion of the particle on the surface (8). By means of

(9) one of the component displacements δx, δy, δz can be eliminated between the two equations; the remaining two displacements being arbitrary, the two equations of motion are found by equating to zero the coefficients of these two displacements.

348. To perform this elimination systematically *the method of indeterminate multipliers* may be used as follows. Multiplying the conditional equation (9) by an indeterminate multiplier λ and adding the resulting equation to (7) we find:

$$(-m\ddot{x} + X + \lambda\varphi_x)\delta x + (-m\ddot{y} + Y + \lambda\varphi_y)\delta y \\ + (-m\ddot{z} + Z + \lambda\varphi_z)\delta z = 0.$$

The arbitrary multiplier λ can be selected so as to make the coefficient of any one of the three displacements vanish; the other two displacements being arbitrary, their coefficients must also vanish. Hence the last equation is equivalent to the three equations,

$$m\ddot{x} = X + \lambda\varphi_x, \quad m\ddot{y} = Y + \lambda\varphi_y, \quad m\ddot{z} = Z + \lambda\varphi_z, \tag{10}$$

which, in connection with the given condition (8), are sufficient to determine x, y, z, and λ as functions of t.

349. By comparing (10) with (3), Art. 330, it appears that

$$X' = \lambda\varphi_x, \quad Y' = \lambda\varphi_y, \quad Z' = \lambda\varphi_z,$$

so that the normal reaction is

$$N = \lambda\sqrt{\varphi_x^2 + \varphi_y^2 + \varphi_z^2}. \tag{11}$$

If we combine the equations (10) by the principle of kinetic energy and work, we find

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz + \lambda(\varphi_x dx + \varphi_y dy + \varphi_z dz).$$

Here the elementary work which constitutes the right-hand member contains in general terms depending on the reaction. This is due to the fact that the displacement $ds(dx, dy, dz)$ here used is along the moving or variable surface (8), and not along its position at the time t.

If the surface (8) be fixed we have of course $\varphi_x dx + \varphi_y dy + \varphi_z dz = 0$ so that the equation reduces to

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz.$$

In the general case, since $\varphi_x dx + \varphi_y dy + \varphi_z dz + \varphi_t dt = 0$, the equation of kinetic energy and work can be written

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy + Zdz - \lambda\varphi_t dt. \tag{12}$$

350. Two constraints. If the particle be subject to two conditions

$$\varphi(x, y, z, t) = 0, \quad \psi(x, y, z, t) = 0 \tag{13}$$

it will move along the curve of intersection of the surfaces represented by these equations.

For a displacement δs along the position of this curve at the time t the work of the reaction is again zero so that the general equation (7) holds for such a displacement. To obtain such a displacement we must subject its components δx, δy, δz to the conditions

$$\varphi_x\delta x + \varphi_y\delta y + \varphi_z\delta z = 0, \quad \psi_x\delta x + \psi_y\delta y + \psi_z\delta z = 0. \tag{14}$$

Between the three equations (7) and (14) two of the displacements δx, δy, δz can be eliminated, and the coefficient of the third equated to zero gives the equation of motion along the curve (13).

351. To perform this elimination in a systematic way, multiply (14) by indeterminate multipliers λ, μ and add to (7). In the resulting equation

$$(-m\ddot{x} + X + \lambda\varphi_x + \mu\psi_x)\delta x + (-m\ddot{y} + Y + \lambda\varphi_y + \mu\psi_y)\delta y + (-m\ddot{z} + Z + \lambda\varphi_z + \mu\psi_z)\delta z = 0$$

the arbitrary multipliers λ, μ can be selected so that the coefficients of two of the displacements δx, δy, δz vanish; and then the coefficient of the third must also vanish. Thus we find the three equations of motion,

$$m\ddot{x} = X + \lambda\varphi_x + \mu\psi_x, \quad m\ddot{y} = Y + \lambda\varphi_y + \mu\psi_y, \quad m\ddot{z} = Z + \lambda\varphi_z + \mu\psi_z, \tag{15}$$

which, together with the conditions (13), are sufficient to determine x, y, z, λ, μ as functions of t.

5. Lagrange's equations of motion.

352. Generalized Co-ordinates. To determine the position of a point P in space we may use, instead of the cartesian co-ordinates x, y, z, a large variety of other systems of co-ordinates, *e. g.* polar or spherical, cylindrical (Art. 56, Ex. 9), elliptic (Arts. 408, 411) co-ordinates, etc. Indeed, any three linearly independent functions of x, y, z, say

$$q_1 = q_1(x, y, z), \quad q_2 = q_2(x, y, z), \quad q_3 = q_3(x, y, z),$$

can be taken as such *generalized*, or *lagrangian, co-ordinates* of P, at least within a certain region of space. Each of these functions equated to a constant represents a surface, and the point $P(x, y, z)$ is determined as intersection of the three surfaces.

Solving these equations for x, y, z we find x, y, z as functions of q_1, q_2, q_3. For the sake of generality we shall assume that x, y, z are given as functions of q_1, q_2, q_3, and of the time t:

$$x = x(q_1, q_2, q_3, t), \quad y = y(q_1, q_2, q_3, t), \quad z = z(q_1, q_2, q_3, t), \tag{16}$$

so that the new system of co-ordinates is a moving or variable system.

By using such generalized co-ordinates and introducing the kinetic energy T and its derivatives the equations of motion of a particle with or without constraints can be put into a remarkably compact form which was first devised by Lagrange for the general equations of motion of a system of n particles (comp. Chap. XX).

353. Free Particle. By multiplying the ordinary equations of motion

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z$$

by $\partial x/\partial q_1$, $\partial y/\partial q_1$, $\partial z/\partial q_1$ and adding we find

$$m\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right) = X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1}.$$

The right-hand member we shall denote briefly by Q_1:

$$Q_1 = X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1};$$

this Q_1 may be called the *generalized force* corresponding to the co-ordinate q_1 (comp. Art. 354).

The main point lies in the transformation of the left-hand member. Consider the first term in the parenthesis; by the formula for the differentiation of a product we have the identity

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1}.$$

Treating the other two terms in the same way we find that our equation can be written:

$$m\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1} + \dot{y}\frac{\partial y}{\partial q_1} + \dot{z}\frac{\partial z}{\partial q_1}\right) - m\left(\dot{x}\frac{\partial \dot{x}}{\partial q_1} + \dot{y}\frac{\partial \dot{y}}{\partial q_1} + \dot{z}\frac{\partial \dot{z}}{\partial q_1}\right) = Q_1, \qquad (17)$$

where the second term is evidently the q_1-derivative of the kinetic energy

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

To interpret the first term observe that the equations (16) give

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3;$$

hence, if we regard $\dot{x}$ as a function of q_1, q_2, q_3, $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$, t, we have

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2}, \quad \frac{\partial \dot{x}}{\partial \dot{q}_3} = \frac{\partial x}{\partial q_3}.$$

Similar relations hold of course for y and z. We can therefore in the first term of (17) replace $\partial x/\partial q_1$, $\partial y/\partial q_1$, $\partial z/\partial q_1$ by $\partial \dot{x}/\partial \dot{q}_1$, $\partial \dot{y}/\partial \dot{q}_1$, $\partial \dot{z}/\partial \dot{q}_1$; and then it appears that this term is equal to the time-derivative of the $\dot{q}_1$-derivative of T. Thus (17) becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1.$$

By multiplying the ordinary equations of motion by the derivatives of x, y, z with respect to q_2 and q_3 we obtain two similar equations. Thus **Lagrange's equations of motion** for a free particle are:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = Q_2, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_3} - \frac{\partial T}{\partial q_3} = Q_3. \quad (18)$$

354. If there exists a force-function U for the forces X, Y, Z, *i. e.* if

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z},$$

we have

$$Q_1 = \frac{\partial U}{\partial x}\frac{\partial x}{\partial q_1} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q_1} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q_1} = \frac{\partial U}{\partial q_1},$$

and similarly

$$Q_2 = \frac{\partial U}{\partial q_2}, \quad Q_3 = \frac{\partial U}{\partial q_3}.$$

In this case one of the three equations (18) can be replaced by the equation of kinetic energy and work

$$T = U + h,$$

where h is a constant.

355. Constrained particle. In the case of one constraint,

$$\varphi(x, y, z, t) = 0,$$

the position of the particle on this surface is determined by two co-ordinates q_1, q_2; and by applying the process of Art. 353 to the equations (10), Art. 348, we find the two equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = Q_2. \tag{18'}$$

For, the coefficients of λ in the right-hand members, viz.

$$\varphi_x \frac{\partial x}{\partial q_1} + \varphi_y \frac{\partial y}{\partial q_1} + \varphi_z \frac{\partial z}{\partial q_1}, \quad \varphi_x \frac{\partial x}{\partial q_2} + \varphi_y \frac{\partial y}{\partial q_2} + \varphi_z \frac{\partial z}{\partial q_2},$$

are zero since the particle moves on the surface $\varphi = 0$.

Similarly, in the case of two constraints,

$$\varphi(x, y, z, t) = 0, \quad \psi(x, y, z, t) = 0,$$

the position of the particle on the curve represented by these equations is determined by a single co-ordinate q, and the equation of motion is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q. \tag{18''}$$

It is obtained from the equations (15), Art. 351, by the process of Art. 353. The coefficient of λ, viz.

$$\varphi_x \frac{\partial x}{\partial q} + \varphi_y \frac{\partial y}{\partial q} + \varphi_z \frac{\partial z}{\partial q},$$

vanishes since it is proportional to the cosine of the angle made at the instant considered by the tangent to the constraining curve with the normal to the surface $\varphi = 0$; similarly for the coefficient of μ.

The equations (18′), (18″) are sometimes distinguished from the equations (10), (15) as *Lagrange's equations of the second kind*, the forms (10), (15) being also due to Lagrange.

CHAPTER XV.

THE EQUATIONS OF MOTION OF A FREE RIGID BODY.

356. In kinetics it is convenient to think of a *rigid body* primarily as a finite number of particles (Art. 156) connected by a rigid framework without mass. The *rigidity* then consists on the one hand, in the invariability of the distances of the particles, on the other in the assumption (Art. 197) that a force applied to the rigid body, *i. e.* to any one of the particles, can be imagined applied at any point of its line of action.

357. Consider any one particle m of the body and let it be cut loose from the other particles; that is, let the members of the framework that attach it to the body be replaced by tensions or pressures. These *internal forces,* together with the *external forces* that may happen to be applied at our particle, will have a resultant F. The equation of motion of this particle is therefore

$$mj = F,$$

or, resolving along fixed rectangular axes,

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z. \tag{1}$$

Notice particularly that the components X, Y, Z of F contain not only the given external, but also the unknown internal, forces.

358. Such a set of three equations can be written down for each particle; hence, if the body consists of n particles, there would be in all $3n$ equations.

The number of conditions expressing the invariability of the distances between n particles is $3n - 6$. For if there

were but 3 particles, the number of independent conditions would evidently be 3; for every additional particle, 3 additional conditions are required. Hence, the total number of conditions is $3 + 3(n - 3) = 3n - 6$.

It follows that if a rigid body be subject to no other constraining conditions, the number of its equations of motion must be $3n - (3n - 6) = 6$. Hence, *a free rigid body has six independent equations of motion* (comp. Art. 231). p. 178

359. The six equations of motion of the rigid body can be obtained as follows.

Imagine the equations (1) written down for every particle, and add the corresponding equations. This gives the first 3 of the 6 equations of motion:

$$\Sigma m\ddot{x} = \Sigma X, \quad \Sigma m\ddot{y} = \Sigma Y, \quad \Sigma m\ddot{z} = \Sigma Z. \qquad (2)$$

It is important to notice that the internal reactions between the particles which make the body rigid occur in pairs of equal and opposite forces, and form, therefore, a system which is in equilibrium by itself. This may be regarded as an assumption which should be included in the definition of the rigid body. Hence, while these internal forces enter into the equations (1), they do not appear in the equations (2). The right-hand members of these equations (2) represent therefore the components R_x, R_y, R_z of the resultant R of all the external forces acting on the body. The left-hand members can be written in the form $d(\Sigma m\dot{x})/dt$, $d(\Sigma m\dot{y})/dt$, $d(\Sigma m\dot{z})/dt$: these are the time-derivatives of the sums of the linear momenta of all the particles parallel to the axes. The equations (2) can therefore be written in the form

$$\frac{d}{dt}\Sigma m\dot{x} = R_x, \quad \frac{d}{dt}\Sigma m\dot{y} = R_y, \quad \frac{d}{dt}\Sigma m\dot{z} = R_z. \qquad (2')$$

The axes of co-ordinates are arbitrary. Hence, if we agree to

call *linear momentum of the body in any direction* the algebraic sum of the linear momenta of all the particles in that direction, the equations (2′) express the proposition that *the rate at which the linear momentum of a rigid body in any direction changes with the time is equal to the sum of the components of all the external forces in that direction.*

360. Let us now combine the second and third of the equations (1) by multiplying the former by z, the latter by y, and subtracting the former from the latter. If this be done for each particle, and the resulting equations be added, we find $\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY)$. Similarly, we can proceed with the third and first, and with the first and second of the equations (1). The result is:

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY), \quad \Sigma m(z\ddot{x} - x\ddot{z}) = \Sigma(zX - xZ),$$
$$\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX). \tag{3}$$

Here again the internal forces disappear in the summation, so that the right-hand members are the components H_x, H_y, H_z, of the vector H of the resultant couple, found by reducing all the external forces for the origin of co-ordinates. The left-hand members are the components of the resultant couple of the effective forces for the same origin.

We can also say that the right-hand members are the sums of the moments of the external forces about the co-ordinate axes (Art. 229), while the left-hand members represent the moments of the effective forces about the same axes. The latter quantities are exact derivatives, as shown in Art. 279. The equations (3) can therefore be written in the form

$$\frac{d}{dt}\Sigma m(y\dot{z} - z\dot{y}) = H_x, \quad \frac{d}{dt}\Sigma m(z\dot{x} - x\dot{z}) = H_y,$$
$$\frac{d}{dt}\Sigma m(x\dot{y} - y\dot{x}) = H_z. \tag{3′}$$

As explained in Art. 279, the quantity $m(y\dot{z} - z\dot{y})$ is called the *angular momentum* (or the moment of momentum) of the particle m about the axis of x. We shall now agree to call the quantity $\Sigma m(y\dot{z} - z\dot{y})$ the *angular momentum of the body* about the axis of x, just as $\Sigma m\dot{x}$ is the linear momentum of the body along this axis; and similarly for the other axes. The meaning of the equations (3′) can then be stated as follows: *The rate at which the angular momentum of a rigid body about any axis changes with the time is equal to the sum of the moments of all the external forces about this line.*

The equations (2) and (3), or (2′) and (3′), are the **six equations of motion of the rigid body.** The three equations (2) or (2′) may be called the *equations of linear momentum,* while (3) or (3′) are the *equations of angular momentum.*

361. If, as in Art. 280, we imagine the angular momentum of each particle represented by a vector drawn from the origin of co-ordinates, the geometric sum, or resultant, of these vectors is a vector h which represents the angular momentum of the body about the origin; and its components h_x, h_y, h_z along the axes are the angular momenta $\Sigma m(y\dot{z} - z\dot{y})$, $\Sigma m(z\dot{x} - x\dot{z})$, $\Sigma m(x\dot{y} - y\dot{x})$ of the body about these axes. The equations (3′) can then be written in the simple form

$$\frac{dh_x}{dt} = H_x, \quad \frac{dh_y}{dt} = H_y, \quad \frac{dh_z}{dt} = H_z; \qquad (3'')$$

and these equations are together equivalent to the single vector equation

$$\frac{dh}{dt} = H.$$

362. The equations of linear momentum, (2) or (2′), admit of a further simplification, owing to the fundamental property

of the centroid. By Art. 159, the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centroid satisfy the relations

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my, \quad M\bar{z} = \Sigma mz,$$

where $M = \Sigma m$ is the whole mass of the body. Differentiating these equations, we find

$$M\dot{\bar{x}} = \Sigma m\dot{x}, \quad M\dot{\bar{y}} = \Sigma m\dot{y}, \quad M\dot{\bar{z}} = \Sigma m\dot{z},$$

and

$$M\ddot{\bar{x}} = \Sigma m\ddot{x}, \quad M\ddot{\bar{y}} = \Sigma m\ddot{y}, \quad M\ddot{\bar{z}} = \Sigma m\ddot{z},$$

where $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are the components of the velocity $\bar{v}$, and $\ddot{\bar{x}}$, $\ddot{\bar{y}}$, $\ddot{\bar{z}}$ those of the acceleration $\bar{j}$, of the centroid.

The equations (2) or (2′) can therefore be reduced to the form

$$M\ddot{\bar{x}} \equiv \frac{d}{dt} M\dot{\bar{x}} = R_x, \quad M\ddot{\bar{y}} \equiv \frac{d}{dt} M\dot{\bar{y}} = R_y, \quad M\ddot{\bar{z}} \equiv \frac{d}{dt} M\dot{\bar{z}} = R_z, \qquad (2'')$$

whence

$$M\bar{j} = \frac{d}{dt} M\bar{v} = R;$$

i. e. if the whole mass of the body be regarded as concentrated at the centroid, the effective force of the centroid, or the time-rate of change of its momentum, is equal to the resultant of all the external forces. It follows that *the centroid of a rigid body moves as if it contained the whole mass, and all the external forces were applied at this point parallel to their original directions.*

363. If, in particular, the resultant R vanish (while there may be a couple H acting on the body), we have by (2″) $\bar{j} = 0$; hence $\bar{v} =$ const.; *i. e. if the resultant force be zero the centroid moves uniformly in a straight line.*

This proposition, which can also be expressed by saying that if $R = 0$, the momentum $M\bar{v}$ of the centroid remains constant, or, using the form (2′) of the equations of motion, that the linear momentum of the body in any direction is constant, is known as the **principle of the conservation of linear momentum,** or the principle of the conservation of the motion of the centroid.

364. Let us next consider the equations of angular momentum, (3) or (3′). To introduce the properties of the centroid, let us put $x - \bar{x} = \xi$, $y - \bar{y} = \eta$, $z - \bar{z} = \zeta$, so that ξ, η, ζ are the co-ordinates of the point (x, y, z) with respect to parallel axes through the centroid. The substitution of $x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$ and their derivatives in the expression $y\dot{z} - z\dot{y}$ gives

$$y\dot{z} - z\dot{y} = \bar{y}\dot{\bar{z}} - \bar{z}\dot{\bar{y}} + \bar{y}\dot{\zeta} - \bar{z}\dot{\eta} + \eta\dot{\bar{z}} - \zeta\dot{\bar{y}} + \eta\dot{\zeta} - \zeta\dot{\eta}.$$

To form $\Sigma m(y\dot{z} - z\dot{y})$ we must multiply by m and sum throughout the body; in this summation, $\bar{y}, \bar{z}, \dot{\bar{y}}, \dot{\bar{z}}$ are constant and by the property of the centroid, $\Sigma m\eta = 0$, $\Sigma m\zeta = 0$, $\Sigma m\dot{\eta} = 0$, $\Sigma m\dot{\zeta} = 0$. Hence we find

$$\Sigma m(y\dot{z} - z\dot{y}) = \Sigma m(\eta\dot{\zeta} - \zeta\dot{\eta}) + M(\bar{y}\dot{\bar{z}} - \bar{z}\dot{\bar{y}}).$$

The second term in the right-hand member is the angular momentum of the centroid about the axis of x (the whole mass M of the body being regarded as concentrated at this point), while the first term is the angular momentum of the body (in its motion relatively to the centroid) about a parallel to the axis of x, drawn through the centroid.

Similar relations hold for the angular momenta about the axes of y and z; and as these axes are arbitrary, we conclude that *the angular momentum of a rigid body about any line is equal to its angular momentum about a parallel through the*

centroid plus the angular momentum of the centroid about the former line.

365. Differentiating the above expression, we find

$$\frac{d}{dt}\Sigma m(y\dot{z} - z\dot{y}) = \frac{d}{dt}\Sigma m(\eta\dot{\zeta} - \zeta\dot{\eta}) + M(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}).$$

The first of the equations (3′) can therefore be written

$$\frac{d}{dt}\Sigma m(\eta\dot{\zeta} - \zeta\dot{\eta}) + M(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}) = H_x.$$

Now, if at any time t the centroid were taken as origin, so that $\bar{y} = 0$, $\bar{z} = 0$, this equation would reduce to the form

$$\frac{d}{dt}\Sigma m(\eta\dot{\zeta} - \zeta\dot{\eta}) = H_x,$$

which is entirely independent of the co-ordinates of the centroid. On the other hand, wherever the origin is taken, if the centroid were a fixed point, the same equation would be obtained.

Similar considerations apply of course to the other two equations (3′). It follows that *the motion of a rigid body relative to the centroid is the same as if the centroid were fixed.*

366. If, in particular, the resultant couple H be zero for any particular origin O (which will be the case not only when all external forces are zero, but whenever the directions of all the forces pass through the point O), the equations (3′) can be integrated and give

$$\Sigma m(y\dot{z} - z\dot{y}) = C_1, \quad \Sigma m(z\dot{x} - x\dot{z}) = C_2, \quad \Sigma m(x\dot{y} - y\dot{x}) = C_3, \tag{4}$$

where C_1, C_2, C_3 are constants of integration. Hence, *if the external forces pass through a fixed point, the angular momentum of the body about any line through this point is constant; if there are no external forces, the angular momentum is constant for*

any line whatever. This is the **principle of the conservation of angular momentum.**

367. Taking the equations of angular momentum in the form (3″) we find when $H = 0$:

$$h_x = C_1, \quad h_y = C_2, \quad h_z = C_3, \tag{4'}$$

and hence the vector h (Fig. 70, Art. 280) remains constant in magnitude and direction. The term *principle of the conservation of areas* which is often used instead of principle of the conservation of angular momentum is less appropriate. In the case of the single particle, where $h_x = m(y\dot{z} - z\dot{y})$, etc., the vector of angular momentum h is simply $2m$ times the vector representing the sectorial velocity; but in the case of the rigid body, to form the vector of angular momentum h we have to multiply the sectorial velocity of each particle by twice its mass and add these "weighted" sectorial velocities geometrically.

In the study of the motion of the rigid body with a fixed point where the vector h is of primary importance it has been called the **impulse,** or impulse-vector. Our principle then means that *whenever for any point O the resultant couple H is zero the impulse remains a constant vector:*

$$h = C.$$

The direction of h is then called the *invariable direction;* the plane through O, perpendicular to h,

$$C_1x + C_2y + C_3z = 0,$$

is called *Laplace's invariable plane.*

368. Returning to the general case of the motion of a rigid body under any forces, we may say that the propositions at the end of Arts. 362 and 365 establish the *principle of the independence of the motions of translation and rotation.* Ac-

cording to these propositions the problem of the motion of a rigid body resolves itself into two problems; that of the motion of the centroid and that of the motion of the body about its centroid. The former reduces by Art. 362 to the problem of the motion of a particle, viz. the centroid of the body, with a mass M equal to that of the body, acted upon by all the given external forces transferred parallel to themselves to the centroid.

The latter problem, that of the motion of the body about its centroid, is, by Art. 365, the same as the problem of the motion of a rigid body about a *fixed* point. This important problem is discussed in Chap. XVIII; its solution depends on the equations (3), (3′), or (3″).

369. If the equation of motion (1), Art. 357, of the particle m be multiplied by the components dx, dy, dz of the actual displacement ds of this particle, we find upon adding the equations for all the particles

$$\Sigma m(\ddot{x}dx + \ddot{y}dy + \ddot{z}dz) = \Sigma(Xdx + Ydy + Zdz),$$

where the right-hand member represents the *elementary work of the external forces* since that of the internal forces is zero. The left-hand member, just as in the case of the single particle (Art. 271), is the exact differential of the kinetic energy

$$T = \Sigma\tfrac{1}{2}mv^2 = \Sigma\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

of the body. Hence, integrating, say from $t = 0$ to $t = t$, we find the relation

$$T - T_0 \equiv \Sigma\tfrac{1}{2}mv^2 - \Sigma\tfrac{1}{2}mv_0{}^2 = \int_0^t \Sigma(Xdx + Ydy + Zdz),$$

where the right-hand member represents the work done by the external forces on the body during the time t. This

equation expresses the **principle of kinetic energy and work,** for a free rigid body: *in any motion of the body, the increase of the kinetic energy is equal to the work done by the external forces.*

370. By introducing the co-ordinates of the centroid, *i. e.* by putting $x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$, as in Art. 364, the expression for the kinetic energy assumes the form (since $\Sigma m\dot{\xi} = 0$, $\Sigma m\dot{\eta} = 0$, $\Sigma m\dot{\zeta} = 0$):

$$\begin{aligned} T &= \Sigma\tfrac{1}{2}m(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) + \Sigma\tfrac{1}{2}m(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) \\ &= \tfrac{1}{2}M\bar{v}^2 + \Sigma\tfrac{1}{2}mu^2, \end{aligned}$$

where $\bar{v}$ is the velocity of the centroid and u the *relative* velocity of any particle m with respect to the centroid.

Thus, it appears that *the kinetic energy of a free rigid body consists of two parts, one of which is the kinetic energy of the centroid* (the whole masss being regarded as concentrated at this point), *while the other may be called the relative kinetic energy with respect to the centroid.*

371. By the same substitution the right-hand member of the first equation of Art. 369, *i. e.* the elementary work $\Sigma(Xdx + Ydy + Zdz)$, resolves itself into the two parts

$$(d\bar{x}\Sigma X + d\bar{y}\Sigma Y + d\bar{z}\Sigma Z) + \Sigma(Xd\xi + Yd\eta + Zd\zeta).$$

The first parenthesis contains the work that would be done by all the external forces if they were applied at the centroid; it is therefore equal to the change in the kinetic energy of the centroid, that is, to $d(\frac{1}{2}M\bar{v}^2)$. The equation of kinetic energy reduces, therefore, to the following

$$d\Sigma(\tfrac{1}{2}mu^2) = \Sigma(Xd\xi + Yd\eta + Zd\zeta);$$

in other words, *the principle of kinetic energy holds for the relative motion with respect to the centroid.*

372. Impulses. The equations determining the effect of a system of impulses on a rigid body are readily obtained from the general equations of motion (2) and (3). We shall denote the impulse of a force F by $\underset{\cdot}{F}$. It will be remembered that the impulse $\underset{\cdot}{F}$ of a force F is its time integral (Art. 172); *i. e.*

$$\underset{\cdot}{F} = \int_t^{t'} F dt.$$

We confine ourselves to the case when $t' - t$ is very small and F very large, in which case the action of the impulsive force F is measured by its impulse $\underset{\cdot}{F}$.

If all the forces acting on a rigid body are of this nature, and the impulses of X, Y, Z during the short interval $t' - t$ be denoted by $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$, the integration of the equations (2) from $t = t$ to $t = t'$ gives

$$\Sigma m(\dot{x}' - \dot{x}) = \Sigma \underset{\cdot}{X}, \quad \Sigma m(\dot{y}' - \dot{y}) = \Sigma \underset{\cdot}{Y}, \quad \Sigma m(\dot{z}' - \dot{z}) = \Sigma \underset{\cdot}{Z}, \quad (5)$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ denote the velocities of the particle m at the time t just before the impulse, and $\dot{x}'$, $\dot{y}'$, $\dot{z}'$ those at the time t' just after the action of the impulse.

Similarly the equations (3) give

$$\begin{aligned} \Sigma m[y(\dot{z}' - \dot{z}) - z(\dot{y}' - \dot{y})] &= \Sigma(y\underset{\cdot}{Z} - z\underset{\cdot}{Y}), \\ \Sigma m[z(\dot{x}' - \dot{x}) - x(\dot{z}' - \dot{z})] &= \Sigma(z\underset{\cdot}{X} - x\underset{\cdot}{Z}), \qquad (6) \\ \Sigma m[x(\dot{y}' - \dot{y}) - y(\dot{x}' - \dot{x})] &= \Sigma(x\underset{\cdot}{Y} - y\underset{\cdot}{X}). \end{aligned}$$

373. In determining the effect on a rigid body of a system of such impulses, any ordinary forces acting on the body at the same time are neglected because the changes of velocity produced by them during the very short time $t' - t$ are small in comparison with the changes of velocity $\dot{x}' - \dot{x}$, $\dot{y}' - \dot{y}$, $\dot{z}' - \dot{z}$ produced by the impulses. If the impulse $\underset{\cdot}{F}$ of an impulsive force F be defined as the limit of the integral

$\int_t^{t'} F dt$ when $t' - t$ approaches zero and F approaches infinity, it is strictly true that the effect of ordinary forces can be neglected when impulsive forces act on the body.

If the rigid body be originally at rest, it will be convenient to denote by $\dot{x}$, $\dot{y}$, $\dot{z}$ the components of the velocity of the particle m just after the action of the impulses. We may also denote by $\underset{\cdot}{R}$ the resultant of all the impulses, by $\underset{\cdot}{H}$ the resultant impulsive couple for the reduction to the origin of co-ordinates, and mark the components of $\underset{\cdot}{R}$ and $\underset{\cdot}{H}$ by subscripts, as in the case of forces. With these notations the effect of a system of impulses on a body at rest is given by the equations

$$\Sigma m\dot{x} = \underset{\cdot}{R}_x, \quad \Sigma m\dot{y} = \underset{\cdot}{R}_y, \quad \Sigma m\dot{z} = \underset{\cdot}{R}_z, \tag{5'}$$

$$\Sigma m(y\dot{z} - z\dot{y}) = \underset{\cdot}{H}_x, \quad \Sigma m(z\dot{x} - x\dot{z}) = \underset{\cdot}{H}_y, \quad \Sigma m(x\dot{y} - y\dot{x}) = \underset{\cdot}{H}_z. \tag{6'}$$

In the equations (5′) we have, of course, $\Sigma m\dot{x} = M\dot{\bar{x}}$, $\Sigma m\dot{y} = M\dot{\bar{y}}$, $\Sigma m\dot{z} = M\dot{\bar{z}}$, where $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ are the components of the velocity of the centroid, and M is the mass of the body; *i. e. the momentum of the centroid is equal to the resultant impulse.* The meaning of the equations (6′) can be stated by saying that *the angular momentum of the body about any axis is equal to the moment of all the impulses about the same axis.*

CHAPTER XVI.

MOMENTS OF INERTIA AND PRINCIPAL AXES.

1. Introduction.

374. As will be shown in Chapters XVII and XVIII, the rotation of a rigid body about any axis depends not only on the forces acting on the body, but also on the way in which the mass is distributed throughout the body. This distribution of mass is characterized by the position of the centroid and by that of certain lines in the body called *principal axes.*

It has been shown in Art. 159 that the centroid of a system of particles is found by determining the *moments,* or more precisely, the *moments of the first order,* Σmx, Σmy, Σmz, of the system with respect to the co-ordinate planes, *i. e.* the sums of all mass-particles m each multiplied by its distance from the co-ordinate plane.

The principal axes of a system of particles can be found by determining the *moments of the second order,* Σmx^2, Σmy^2, Σmz^2, Σmyz, Σmzx, Σmxy of the system with respect to the same planes. We proceed, therefore, to study the theory of such moments.

375. If in a rigid body the mass m of each particle be multiplied by the square of its distance r from a given point, plane, or line, the sum

$$\Sigma mr^2 = m_1r_1^2 + m_2r_2^2 + \cdots,$$

extended over the whole body, is called the *quadratic moment,* or, more commonly, the **moment of inertia** of the body for that point, plane, or line.

If the body is not composed of discrete particles, but forms a continuous mass of one, two, or three dimensions, this mass can be resolved into elements of mass dm, and the sum Σmr^2 becomes a single, double, or triple integral $\int r^2 dm$.

Expressions of the form Σmr_1r_2, or $\int r_1r_2dm$, where r_1, r_2 are the distances of m or of dm from two planes (usually at right angles), are called *moments of deviation* or **products of inertia.**

376. The determination of the moment of inertia of a continuous mass is a mere problem of integration; the methods are similar to those for finding the moments of mass of the first order required for determining centroids, the only difference being that each element of mass must be multiplied by the square, instead of the first power, of the distance.

A moment of inertia is not a directed quantity; it is not a vector, but a scalar; indeed, it is a positive quantity, provided the masses are all positive, as we shall here assume.

If the mass is homogeneous, the density appears merely as a constant factor; as the density in this case can be regarded as $= 1$, it is customary to speak of moments of inertia of volumes, areas, and lines.

The moment of inertia of any number of bodies or masses for any given point, plane, or line is obviously the sum of the moments of inertia of the separate bodies or masses for the same point, plane, or line.

377. The moment of inertia Σmr^2 of any body whose mass is $M = \Sigma m$ can always be expressed in the form

$$\Sigma mr^2 = M \cdot r_0^2,$$

where r_0 is a length called the **radius of inertia,** arm of inertia, or *radius of gyration*. This length r_0 is evidently a kind of average value of the distances r, its value being intermediate between the greatest r' and least r'' of these distances r. For

we have $\Sigma mr'^2 \geqq \Sigma mr^2 \geqq \Sigma mr''^2$, or, since $\Sigma mr'^2 = Mr'^2$, $\Sigma mr^2 = Mr_0^2$, $\Sigma mr''^2 = Mr''^2$,

$$r' \geqq r_0 \geqq r''.$$

378. As an example, let us determine the moment of inertia of a *homogeneous rectilinear segment* (straight rod or wire of constant cross-section and density) for its middle point (or what amounts to the same thing, for a line or plane through this point at right angles to the segment).

Let l be the length of the rod (Fig. 80), O its middle point, ρ'' its density (*i. e.* the mass of unit length), x the distance OP of any element $dm = \rho'' dx$ from the middle point. Observing that the moment of inertia for O of the whole rod AB is the sum of the moments of inertia of the halves AO and OB, and that the moments of inertia of these halves are equal, we have, for the moment of inertia I of AB,

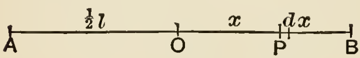

Fig. 80.

$$I = 2\int_0^{\frac{1}{2}l} x^2 \cdot \rho'' dx = \frac{1}{12}\rho'' l^3,$$

and for the radius of inertia r_0, since the whole mass is $M = \rho'' l$,

$$r_0^2 = \frac{I}{M} = \frac{1}{12} l^2.$$

379. Exercises.

Determine the radius of inertia in the following cases. When nothing is said to the contrary, the masses are supposed to be homogeneous.

(1) Segment of straight line of length l, for perpendicular through one end.

(2) Rectangular area of length l and width h: (a) for the side h; (b) for the side l; (c) for a line through the centroid parallel to the side h; (d) for a line through the centroid parallel to the side l.

(3) Triangular area of base b and height h, for a line through the vertex parallel to the base.

(4) Square of side a, for a diagonal.

(5) Regular hexagon of side a, for a diagonal.

(6) Right cylinder or prism of height h, for the plane bisecting the height at right angles.

(7) Segment of straight line of length l, for one end, when the density is proportional to the nth power of the distance from this end. Deduce from this: (a) the result of Ex. (1); (b) that of Ex. (3); (c) the radius of inertia of a homogeneous pyramid or cone (right or oblique) of height h, for a plane through the vertex parallel to the base.

(8) Circular area (plate, disk, lamina) of radius a, for any diameter.

(9) Circular line (wire) of radius a, for a diameter.

(10) Solid sphere, for a diametral plane.

(11) Solid ellipsoid, for the three principal planes.

(12) Area of ring bounded by concentric circles of radii a_1, a_2, for a diameter.

380. The moment of inertia of any mass M *for a point* can easily be found if the moments of inertia of the same mass are known for any line passing through the point, and for the plane through the point perpendicular to the line. Let O (Fig. 81) be the point, l the line, π the plane; r, q, p the perpendicular distances of any particle of mass m from O, l, π, respectively. Then we have, evidently, $r^2 = q^2 + p^2$. Hence, multiplying by m, and summing over the whole mass M,

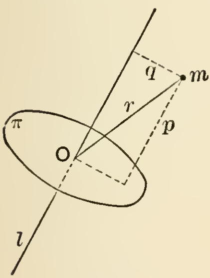

Fig. 81.

$$\Sigma mr^2 = \Sigma mq^2 + \Sigma mp^2; \tag{1}$$

or, putting $\Sigma mr^2 = Mr_0^2$, $\Sigma mq^2 = Mq_0^2$, $\Sigma mp^2 = Mp_0^2$, where r_0, q_0, p_0 are the radii of inertia for O, l, π,

$$r_0^2 = q_0^2 + p_0^2. \tag{1'}$$

381. The moment of inertia of any mass M *for a line* is equal to the sum of the moments of inertia of the same mass for any two rectangular planes passing through the line. Thus, in particular, the moment of inertia for the axis of x in a rectangular system of co-ordinates is equal to the sum of

the moments of inertia for the zx-plane and xy-plane. This follows at once by considering that the square of the distance of any point from the line is equal to the sum of the squares of the distances of the same point from the two planes. Thus, if q be the distance of any point (x, y, z) from the axis of x, we have $q^2 = y^2 + z^2$; whence

$$\Sigma mq^2 = \Sigma my^2 + \Sigma mz^2.$$

382. It follows, from the last article, that *the moment of inertia I_x of a plane area, for any line perpendicular to its plane*, is

$$I_x = I_y + I_z,$$

if I_y, I_z are the moments of inertia of the area for any two rectangular lines in the plane through the foot of the perpendicular line.

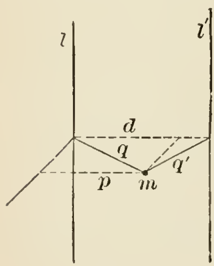

Fig. 82.

383. The problem of *finding the moment of inertia of a given mass for a line l', when it is known for a parallel line l*, is of great importance.

Let Σmq^2 be the moment of inertia of the given mass for the line l (Fig. 82), $\Sigma mq'^2$ that for a parallel line l' at the distance d from l. The distances q, q' of any particle m from l, l' form with d a triangle which gives the relation

$$q'^2 = q^2 + d^2 - 2qd \cos(q, d).$$

Multiplying by m, and summing over the whole mass M, we find

$$\Sigma mq'^2 = \Sigma mq^2 + Md^2 - 2d\Sigma mq \cos(q, d).$$

Now the figure shows that the product $q \cos(q, d)$ in the last term is the distance p of the particle m from a plane

through l at right angles to the plane determined by l and l'. We have, therefore,

$$\Sigma mq'^2 = \Sigma mq^2 + Md^2 - 2d\Sigma mp, \qquad (2)$$

where the last term contains the moment of the first order $\Sigma mp = M\bar{p}$ of the given mass M for the plane just mentioned.

If, in particular, this plane contains the centroid G of the mass M, we have $\Sigma mp = 0$, so that the formula reduces to

$$\Sigma mq'^2 = \Sigma mq^2 + Md^2. \qquad (3)$$

Introducing the radii of inertia q_0', q_0, this can be written

$$q_0'^2 = q_0^2 + d^2. \qquad (3')$$

384. Similar considerations hold for the moments of inertia Σmp^2, $\Sigma mp'^2$ with respect to two parallel planes π, π' at the distance d from each other. We have, in this case, $p' = p - d$; hence,

$$\Sigma mp'^2 = \Sigma mp^2 + Md^2 - 2d\Sigma mp, \qquad (4)$$

and if the plane π contain the centroid G,

$$\Sigma mp'^2 = \Sigma mp^2 + Md^2. \qquad (5)$$

385. Of special importance is the case in which one of the lines (or planes), say $l(\pi)$, contains the centroid. The formulæ (3), (3′), and (5) hold in this case; and if we agree to designate any line (plane) passing through the centroid as a *centroidal* line (plane), our proposition can be expressed as follows: *The moment of inertia for any line* (*plane*) *is found from the moment of inertia for the parallel centroidal line* (*plane*) *by adding to the latter the product* Md^2 *of the whole mass into the square of the distance of the lines* (*planes*).

It will be noticed that of all parallel lines (planes) the centroidal line (plane) has the least moment of inertia.

386. Exercises.

Determine the radius of inertia of the following homogeneous masses:

(1) Rectangular plate of length l and width h, for a centroidal line perpendicular to its plane.

(2) Area of equilateral triangle of side a: (a) for a centroidal line parallel to the base; (b) for an altitude; (c) for a centroidal line perpendicular to its plane.

(3) Circular disk of radius a: (a) for a tangent; (b) for a line through the center perpendicular to the plane of the disk; (c) for a perpendicular to its plane through a point in the circumference.

(4) Solid sphere, for a diameter.

(5) Area of ring bounded by concentric circles of radii a_1, a_2, for a line through the center perpendicular to the plane of the ring.

(6) Right circular cylinder, of radius a and height h: (a) for its axis; (b) for a generating line; (c) for a centroidal line in the middle cross-section.

(7) By Ex. (3) (b), the moment of inertia of the area of a circle of radius a, for its *axis* (*i. e.* the perpendicular to its plane, passing through the center), is $I = \frac{1}{2}\pi a^4$. Differentiating with respect to a, we find:

$$\frac{dI}{da} = 2\pi a^3 = 2\pi a \cdot a^2;$$

hence, approximately for small Δa:

$$\Delta I = 2\pi a^3 \Delta a = 2\pi a \Delta a \cdot a^2.$$

This is the moment of inertia of the thin ring, of thickness Δa, for its axis. (Comp. Ex. (5).)

If the constant *surface* density (Art. 155) of the circle be ρ', we have $I = \frac{1}{2}\rho'\pi a^4$; hence

$$\Delta I = 2\pi a \rho' \Delta a \cdot a^2,$$

where $\rho'\Delta a$ is the *linear* density ρ'' of the ring.

(8) Apply the method of Ex. (7) to derive from Ex. (4) the moment of inertia of a thin spherical shell, of radius a and thickness Δa, for a diameter.

(9) Area of ellipse: (a) for the major axis; (b) for the minor axis; (c) for the perpendicular to its plane through the center.

(10) Solid ellipsoid, for each of the three axes.

(11) Wire bent into an equilateral triangle of side a, for a centroidal line at right angles to the plane of the triangle.

(12) Paraboloid of revolution, bounded by the plane through the focus at right angles to the axis, for the axis.

(13) Anchor-ring, produced by the revolution of a circle of radius a about a line in its plane at the distance $b(> a)$ from the center, for the axis of revolution.

2. Ellipsoids of inertia.

387. The moments of inertia of a given mass for the different lines of space are not independent of each other. Several examples of this have already been given. It has been shown, in particular (Art. 383), that if the moment of inertia be known for any line, it can be found for any parallel line. It follows that if the moments be known for all lines through any given point, the moments for all lines of space can be found. We now proceed to study the relations between the moments of inertia for all the lines passing through any given point O.

Let x, y, z be the co-ordinates of any particle m of the mass; and let us denote by A, B, C the moments of inertia of M for the axes of x, y, z; by A', B', C' those for the planes yz, zx, xy; by D, E, F the products of inertia (Art. 375) for the co-ordinate planes; *i. e.* let us put

$$\begin{aligned} A &= \Sigma m(y^2 + z^2), & A' &= \Sigma mx^2, & D &= \Sigma myz, \\ B &= \Sigma m(z^2 + x^2), & B' &= \Sigma my^2, & E &= \Sigma mzx, \\ C &= \Sigma m(x^2 + y^2), & C' &= \Sigma mz^2, & F &= \Sigma mxy. \end{aligned} \tag{6}$$

388. These nine quantities are not independent of each other. We have evidently

$$A = B' + C', \quad B = C' + A', \quad C = A' + B';$$

hence, solving for A', B', C',

$$A' = \tfrac{1}{2}(B + C - A), \quad B' = \tfrac{1}{2}(C + A - B), \quad C' = \tfrac{1}{2}(A + B - C).$$

The moment of inertia for the origin O is

$$\Sigma mr^2 = \Sigma m(x^2 + y^2 + z^2) = A' + B' + C' = \tfrac{1}{2}(A + B + C). \tag{7}$$

389. The moment of inertia I for any line through O can be expressed by means of the six quantities A, B, C, D, E, F; and the moment of inertia I' for any plane through O can be expressed by means of A', B', C', D, E, F.

Let π (Fig. 83) be any plane passing through O; l its normal; α, β, γ the direction cosines of l; and, as before (Art. 380), p, q, r the distances of any point (x, y, z) of the given mass from π, l, and O, respectively. Then, projecting the closed polygon formed by r, x, y, z on the line l, we have

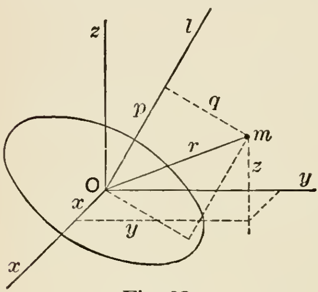

Fig. 83.

$$p = \alpha x + \beta y + \gamma z;$$

hence, squaring, multiplying by m, and summing over the whole mass, we find

$$\Sigma mp^2 = \alpha^2 \Sigma mx^2 + \beta^2 \Sigma my^2 + \gamma^2 \Sigma mz^2 + 2\beta\gamma \Sigma myz + 2\gamma\alpha \Sigma mzx + 2\alpha\beta \Sigma mxy,$$

or, with the notations (6),

$$I' = A'\alpha^2 + B'\beta^2 + C'\gamma^2 + 2D\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta. \quad (8)$$

Thus *the moment of inertia for any plane through the origin is expressed as a homogeneous quadratic function of the direction cosines of the normal of the plane.*

390. The moment of inertia $I = \Sigma mq^2$ for the line l can now be found from equation (1), Art. 380, by substituting for Σmr^2 and Σmp^2 their values from (7) and (8):

$$I = \Sigma mr^2 - I' = A' + B' + C' - I'$$
$$= A'(1-\alpha^2) + B'(1-\beta^2) + C'(1-\gamma^2) - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta,$$

or, since $\alpha^2 + \beta^2 + \gamma^2 = 1$,

$$I = A'(\beta^2 + \gamma^2) + B'(\gamma^2 + \alpha^2) + C'(\alpha^2 + \beta^2) \\ - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta$$
$$= \alpha^2(B' + C') + \beta^2(C' + A') + \gamma^2(A' + B') \\ - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta;$$

hence, finally, applying the relations of Art. 388,

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta. \quad (9)$$

The moment of inertia for any line through the origin is, therefore, also *a homogeneous quadratic function of the direction cosines of the line.*

391. These results suggest a geometrical interpretation. Imagine an arbitrary length $OP = \rho$ laid off from the origin O on the line l whose direction cosines are α, β, γ; the coordinates of the extremity P of this length will be $x = \rho\alpha$, $y = \rho\beta$, $z = \rho\gamma$. Now, if equation (9) be multiplied by ρ^2, it assumes the form

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \rho^2 I,$$

which represents a quadric surface provided that ρ be selected for the different lines through O so as to make $\rho^2 I$ constant, say $\rho^2 I = \kappa^2$. Hence, *if on every line l through the origin a length $OP = \rho = \kappa/\sqrt{I}$ be laid off, i. e. a length inversely proportional to the square root of the moment of inertia I for this line l, the points P will lie on the quadric surface*

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \kappa^2.$$

The constant κ^2 may be selected arbitrarily; to preserve the homogeneity of the equation it will be convenient to take it in the form $\kappa^2 = M\epsilon^4$, where ϵ is an arbitrary length.

392. As moments of inertia are essentially positive quantities, the radii vectores of the surface

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = M\epsilon^4 \quad (10)$$

are all real, and the surface is an ellipsoid. It is called the *ellipsoid of inertia*, or the **momental ellipsoid**, of the point O. This point O is the center; the axes of the ellipsoid are called the **principal axes** at the point O; and the moments of inertia for these axes are called the *principal moments of inertia* at the point O. Among these there will evidently be the greatest and least of all the moments of the point O, the greatest moment corresponding to the shortest, the least to the longest axis of the ellipsoid.

It may be observed that, owing to the relations of Art. 388, which show that the sum of any two of the quantities A, B, C is always greater than the third, not every ellipsoid can be regarded as the momental ellipsoid of some mass. An ellipsoid can be a momental ellipsoid only when a triangle can be constructed of the reciprocals of the squares of its semi-axes.

393. If the axes of the ellipsoid (10) be taken as axes of co-ordinates, the equation assumes the form

$$I_1x^2 + I_2y^2 + I_3z^2 = M\epsilon^4, \tag{11}$$

where I_1, I_2, I_3 are the principal moments at the point O.

By Art. 391 we have $\rho^2 = \kappa^2/I = M\epsilon^4/I$; hence $I = M\epsilon^4/\rho^2$. If, therefore, equation (11) be divided by ρ^2, the following simple expression is obtained for finding the moment of inertia, I, for a line whose direction cosines referred to the principal axes are α, β, γ:

$$I = I_1\alpha^2 + I_2\beta^2 + I_3\gamma^2. \tag{12}$$

394. To make use of this form for I, the principal axes at the point O, *i. e.* the axes of the momental ellipsoid (10), must be known. The determination of the axes of an ellipsoid whose equation referred to the center is given is a well-known problem of analytic geometry. It can be solved by considering that the semi-axes are those radii vectores of the surface that are normal to it. The direction cosines of the normal

of any surface $F(x, y, z) = 0$ are proportional to the partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$. If, therefore, the radius vector ρ is a semi-axis, its direction-cosines α, β, γ must be proportional to the partial derivatives of the left-hand member of (10); *i. e.* we must have

$$\frac{Ax - Fy - Ez}{\alpha} = \frac{-Fx + By - Dz}{\beta} = \frac{-Ex - Dy + Cz}{\gamma},$$

or dividing the numerators by ρ,

$$\frac{A\alpha - F\beta - E\gamma}{\alpha} = \frac{-F\alpha + B\beta - D\gamma}{\beta} = \frac{-E\alpha - D\beta + C\gamma}{\gamma}.$$

Denoting the common value of these fractions by I we have

$$\alpha I = A\alpha - F\beta - E\gamma,\quad \beta I = -F\alpha + B\beta - D\gamma,\quad \gamma I = -E\alpha - D\beta + C\gamma;$$

multiplying these equations by α, β, γ and adding we find

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta,$$

which, compared with (9), shows that I is the moment of inertia for the axis (α, β, γ). To obtain it in terms of A, B, C, D, E, F, we write the preceding three equations in the form

$$\begin{aligned} (I - A)\alpha + \quad F\beta + \quad E\gamma &= 0, \\ F\alpha + (I - B)\beta + \quad D\gamma &= 0, \\ E\alpha + \quad D\beta + (I - C)\gamma &= 0, \end{aligned} \tag{13}$$

whence, eliminating α, β, γ, we find I determined by the cubic equation

$$\begin{vmatrix} I - A, & F, & E \\ F, & I - B, & D \\ E, & D, & I - C \end{vmatrix} = 0. \tag{14}$$

The roots of this cubic are the three principal moments I_1, I_2, I_3 of the point O. The direction cosines of the principal axes are then found by substituting successively I_1, I_2, I_3 in (13) and solving for α, β, γ.

In general, the three principal moments of inertia I_1, I_2, I_3 at a point O are different. If, however, two of them are equal, say $I_2 = I_3$, the momental ellipsoid becomes an ellipsoid of revolution about the third, I_1, as axis; and it follows that the moments of inertia for all lines through O in the plane of the two equal axes are equal.

If $I_1 = I_2 = I_3$, the ellipsoid becomes a sphere, and the moments of inertia are the same for all lines passing through O.

395. If the equation of the momental ellipsoid at a point O be of the form $Ax^2 + By^2 + Cz^2 - 2Dyz = M\epsilon^4$, *i. e.* if the two conditions

$$E \equiv \Sigma mzx = 0, \quad F \equiv \Sigma mxy = 0$$

be fulfilled, the axis of x coincides with one of the three axes of the ellipsoid, the surface being symmetrical with respect to the yz-plane. Hence, *if the conditions* $E = 0$, $F = 0$ *are satisfied, the axis of* x *is a principal axis at the origin.* The converse is evidently also true; *i. e.* if a line is a principal axis at one of its points, then, for this point as origin and the line as axis of x, the conditions $\Sigma mzx = 0$, $\Sigma mxy = 0$ must be satisfied.

It is easy to see that if a line be a principal axis at one of its points, say O, it will in general not be a principal axis at any other one of its points. For, taking the line as axis of x and O as origin, we have $\Sigma mzx = 0$, $\Sigma mxy = 0$. If now for a point O' on this line at the distance a from O the line is likewise a principal axis, the conditions

$$\Sigma mz(x - a) = 0, \quad \Sigma m(x - a)y = 0$$

must be fulfilled. These reduce to

$$\Sigma mz = 0, \quad \Sigma my = 0,$$

and show that the line must pass through the centroid. And as for a centroidal line these conditions are satisfied independently of the value of a, it appears that a centroidal principal axis is a principal axis at every one of its points. Hence, *a line cannot be principal axis at more than one of its points unless it pass through the centroid; in the latter case it is a principal axis at every one of its points.*

396. All those lines passing through a given point O for which the moments of inertia have the same value I can be shown to form a cone of the second order whose principal diameters coincide with the axes of the momental ellipsoid at O. This is called an **equimomental cone.** Its equation is obtained by regarding I as constant in equation (12) and introducing rectangular co-ordinates. Multiplying (12) by $\alpha^2 + \beta^2 + \gamma^2 = 1$, we find

$$(I_1 - I)\alpha^2 + (I_2 - I)\beta^2 + (I_3 - I)\gamma^2 = 0;$$

and multiplying by ρ^2, we obtain the equation of the equimomental cone in the form

$$(I_1 - I)x^2 + (I_2 - I)y^2 + (I_3 - I)z^2 = 0. \tag{15}$$

397. A slightly different form of the equations (11), (12), (15) is often more convenient; it is obtained by introducing the three **principal radii of inertia** q_1, q_2, q_3 defined by the relations

$$I_1 = Mq_1^2, \quad I_2 = Mq_2^2, \quad I_3 = Mq_3^2.$$

The equation (11) of the momental ellipsoid at the point O then assumes the form

$$q_1^2x^2 + q_2^2y^2 + q_3^2z^2 = \epsilon^4. \tag{11'}$$

The expression of the radius of inertia q for any line (α, β, γ) through O becomes

$$q^2 = q_1^2\alpha^2 + q_2^2\beta^2 + q_3^2\gamma^2. \tag{12'}$$

Dividing (11′) by the square of the radius vector, ρ^2, and comparing with (12′), we find

$$q = \frac{\epsilon^2}{\rho}, \quad \rho = \frac{\epsilon^2}{q}, \tag{16}$$

as is otherwise apparent from the fundamental property of the momental ellipsoid (Art. 391).

The equation of the equimomental cone for all whose generators the radius of inertia has the value q is obtained from (15) in the form

$$(q_1^2 - q^2)x^2 + (q_2^2 - q^2)y^2 + (q_3^2 - q^2)z^2 = 0. \tag{15'}$$

This cone meets any one of the momental ellipsoids (11′) in points whose radii vectores ρ are all equal; and if we select the arbitrary constant ϵ equal to the radius of inertia q of the generators of the equimomental cone, it follows from (16) that $\rho = q$. This particular ellipsoid has the equation

$$q_1^2x^2 + q_2^2y^2 + q_3^2z^2 = q^4,$$

and its intersection with the equimomental cone (15′) lies on the sphere

$$x^2 + y^2 + z^2 = q^2.$$

In other words, the equimomental cone (15′) passes through the spheroconic in which the ellipsoid meets the sphere. Multiplying the equation of the sphere by q^2 and subtracting it from the equation of the ellipsoid we obtain the equation (15′) of the cone.

The semi-axes of the ellipsoid are q^2/q_1, q^2/q_2, q^2/q_3. If we assume $I_1 > I_2 > I_3$, and hence $q_1 > q_2 > q_3$, q must be $\leqq q^2/q_3$ and $\geqq q^2/q_1$ As long as q is less than the middle semi-axis q^2/q_2 of the ellipsoid, the

axis of the cone coincides with the axis of x; but when $q > q^2/q_2$, the axis of z is the axis of the cone. For $q = q^2/q_2$, *i. e.* $q = q_2$, the cone (15′) degenerates into the pair of planes $(q_1^2 - q_2^2)x^2 - (q_2^2 - q_3^2)z^2 = 0$. These are the planes of the central circular (or *cyclic*) sections of the ellipsoid; they divide the ellipsoid into four wedges, of which one pair contains all the equimomental cones whose axes coincide with the greatest axis of the ellipsoid, while the other pair contains all those whose axes lie along the least axis of the ellipsoid.

398. There is another ellipsoid closely connected with the theory of principal axes; it is obtained from the momental ellipsoid by the process of reciprocation.

About any point O (Fig. 84) as center let us describe a sphere of radius ϵ, and construct for every point P its polar plane π with regard to the sphere. If P describe any surface, the plane π will envelop another surface which is called the *polar reciprocal* of the former surface with regard to the sphere.

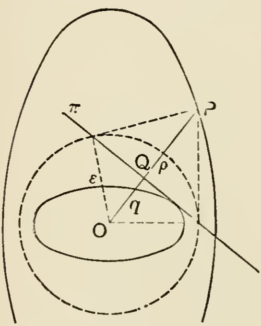

Fig. 84.

Let Q be the intersection of OP with π, and put $OP = \rho$, $OQ = q$; then it appears from the figure that

$$\rho q = \epsilon^2. \tag{16}$$

399. It is easy to see that the polar reciprocal of the momental ellipsoid (11′) with respect to the sphere of radius ϵ is the ellipsoid

$$\frac{x^2}{q_1^2} + \frac{y^2}{q_2^2} + \frac{z^2}{q_3^2} = 1. \tag{17}$$

To prove this it is only necessary to show that the relation (16) is fulfilled for ρ as radius vector of (11′), and q as perpendicular to the tangent plane of (17). Now this tangent plane has the equation

$$\frac{x}{q_1^2} X + \frac{y}{q_2^2} Y + \frac{z}{q_3^2} Z = 1;$$

hence we have for the direction cosines α, β, γ and for the length q of the perpendicular to the tangent plane

$$\frac{\alpha}{x/q_1^2} = \frac{\beta}{y/q_2^2} = \frac{\gamma}{z/q_3^2} = \frac{1}{[x^2/q_1^4 + y^2/q_2^4 + z^2/q_3^4]^{\frac{1}{2}}} = q.$$

These relations give $q_1\alpha = (x/q_1)q$, $q_2\beta = (y/q_2)q$, $q_3\gamma = (z/q_3)q$, whence

$$q_1^2\alpha^2 + q_2^2\beta^2 + q_3^2\gamma^2 = \left(\frac{x^2}{q_1^2} + \frac{y^2}{q_2^2} + \frac{z^2}{q_3^2}\right) q^2 = q^2. \tag{18}$$

For the radius vector ρ of (11′) whose direction cosines α, β, γ are the same as those of q, we have by (11′):

$$\rho^2 = \frac{\epsilon^4}{q_1^2\alpha^2 + q_2^2\beta^2 + q_3^2\gamma^2}.$$

Hence $\rho^2 q^2 = \epsilon^4$; and this is what we wished to prove.

400. The surface (17) has variously been called the *ellipsoid of gyration*, the *ellipsoid of inertia*, the **reciprocal ellipsoid.** We shall adopt the last name. The semi-axes of this ellipsoid are equal to the principal radii of inertia at the point O. The directions of its axes coincide with those of the momental ellipsoid; but the greatest axis of the former coincides with the least of the latter, and *vice versa.*

By comparing the equations (12′) and (18) it will be seen that q is the radius of inertia of the line (α, β, γ) on which it lies. Thus, *while the radius vector $OP = \rho$ of the momental ellipsoid is inversely proportional to the radius of inertia, i. e. $\rho = \epsilon^2/q$, the reciprocal ellipsoid gives the radius of inertia q for a line as the segment cut off on this line by the perpendicular tangent plane.*

401. We are now prepared to determine the moment of inertia for any line in space. Let us construct at the centroid G of the given mass or body both the momental ellipsoid and its polar reciprocal. The former is usually called the **central ellipsoid** of the body; the latter we may call the **fundamental ellipsoid** of the body. As soon as this fundamental ellipsoid

$$\frac{x^2}{q_1^2} + \frac{y^2}{q_2^2} + \frac{z^2}{q_3^2} = 1$$

is known, the moment of inertia of the body for any line whatever can readily be found. For, by Art. 400, the radius of inertia q for any line l_0 passing through the centroid is equal to the segment OQ cut off on the line l_0 by the perpendicular tangent plane of the fundamental ellipsoid; and for any line l not passing through the centroid, the square of the radius of inertia can be determined by first finding the

square of the radius of inertia for the parallel centroidal line l_0, and then, by Art. 385, adding to it the square of the distance d of the centroid from the line l.

402. In the problem of determining the ellipsoids of inertia for a given body at any point, considerations of symmetry are of great assistance.

Suppose a given mass to have a plane of symmetry; then taking this plane as the yz-plane, and a perpendicular to it as the axis of x, there must be, for every particle of mass m, whose co-ordinates are x, y, z, another particle of equal mass m, whose co-ordinates are $-x$, y, z. It follows that the two products of inertia Σmzx and Σmxy both vanish, whatever the position of the other two co-ordinate planes. Hence, any perpendicular to the plane of symmetry is a principal axis at its point of intersection with this plane.

Let the mass have two planes of symmetry at right angles to each other; then taking one as yz-plane, the other as zx-plane, and hence their intersection as axis of x, it is evident that all three products of inertia vanish,

$$\Sigma myz = 0, \qquad \Sigma mzx = 0, \qquad \Sigma mxy = 0,$$

wherever the origin be taken on the intersection of the two planes. Hence, for any point on this intersection, the principal axes are the line of intersection of the two planes of symmetry, and the two perpendiculars to it, drawn in each plane.

If there be three planes of symmetry, their point of intersection is the centroid, and their lines of intersection are the principal axes at the centroid.

403. Exercises.

Determine the principal axes and radii at the centroid, the central and fundamental ellipsoids, and show how to find the moment of inertia for any line, in the following Exercises (1), (2), (3).

(1) Rectangular parallelepiped, the edges being $2a$, $2b$, $2c$. Find also the moments of inertia for the edges and diagonals, and specialize for the cube.

(2) Ellipsoid of semi-axes a, b, c. Determine also the radius of inertia for a parallel l to the shortest axis passing through the extremity of the longest axis.

(3) Right circular cone of height h and radius of base a. Find

first the principal moments at the vertex; then transfer to the centroid.

(4) Determine the momental ellipsoid and the principal axes at a vertex of a cube whose edge is a.

(5) Determine the radius of inertia of a thin wire bent into a circle, for a line through the center inclined at an angle α to the plane of the circle.

(6) A peg-top is composed of a cone of height H and radius a, and a hemispherical cap of the same radius. The pointed end, to a distance h from the vertex of the cone, is made of a material three times as heavy as the rest. Find the moment of inertia for the axis of rotation; specialize for $h = a = \frac{1}{3}H$.

(7) Show that the principal axes at any point P, situated on one of the principal axes of a body, are parallel to the centroidal principal axes, and find their moments of inertia.

(8) For a given body of mass M find the points (*spherical points of inertia*) at which the momental ellipsoid reduces to a sphere.

(9) Determine a homogeneous ellipsoid having the same mass as a given body, and such that its moment of inertia for every line shall be the same as that of the given body.

(10) For a given body M, whose centroidal principal radii are q_1, q_2, q_3, determine three homogeneous straight rods intersecting at right angles, of such lengths $2a$, $2b$, $2c$, and such linear density ρ'', that they have the same mass and the same moment of inertia (for any line) as the given body.

3. Distribution of principal axes in space.

404. It has been shown in the preceding articles how the principal axes can be determined at any particular point. The distribution of the principal axes throughout space and their position at the different points is brought out very graphically by means of the theory of confocal quadrics. It can be shown that the directions of the principal axes at any point are those of the principal diameters of the tangent cone drawn from this point as vertex to the fundamental ellipsoid; or, what amounts to the same thing, they are the directions of the normals of the three quadric surfaces passing through the point and confocal to the fundamental ellipsoid.

In order to explain and prove these propositions it will be necessary to give a short sketch of the theory of confocal conics and quadrics.

405. *Two conic sections are said to be* **confocal** *when they have the same foci.* The directions of the axes of all conics having the same two points S, S' as foci must evidently coincide, and the equation of such conics can be written in the form

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \tag{19}$$

where λ is an arbitrary parameter. For, whatever value may be assigned in this equation to λ, the distance of the center O from either focus will always be $\sqrt{a^2+\lambda-(b^2+\lambda)}=\sqrt{a^2-b^2}$; it is therefore constant.

406. The individual curves of the whole system of confocal conics represented by (19) are obtained by giving to λ any particular value between $-\infty$ and $+\infty$; thus we may speak of the conic λ of the system.

For $\lambda = 0$ we have the so-called fundamental conic $x^2/a^2 + y^2/b^2 = 1$; this is an ellipse. To fix the ideas let us assume $a > b$. For all values of $\lambda > -b^2$, *i. e.* as long as $-b^2 < \lambda < \infty$, the conics (19) are ellipses, beginning with the rectilinear segment SS' (which may be regarded as a degenerated ellipse $\lambda = -b^2$ whose minor axis is 0), expanding gradually, passing through the fundamental ellipse $\lambda = 0$, and finally verging into a circle of infinite radius for $\lambda = \infty$.

It is thus geometrically evident that through every point in the plane will pass one, and only one, of these ellipses.

407. Let us next consider what the equation (19) represents when λ is algebraically less than $-b^2$. The values of λ that are $< -a^2$ give imaginary curves, and are of no importance for our purpose. But as long as $-a^2 < \lambda < -b^2$, the curves are hyperbolas. The curve $\lambda = -b^2$ may now be regarded as a degenerated hyperbola collapsed into the two rays issuing in opposite directions from S and S' along the line SS'. The degenerated ellipse together with this degenerated hyperbola thus represents the whole axis of x.

As λ decreases, the hyperbola expands, and finally, for $\lambda = -a^2$, verges into the axis of y, which may be regarded as another degenerated hyperbola.

The system of confocal hyperbolas is thus seen to cover likewise the whole plane so that one, and only one, hyperbola of the system passes through every point of the plane.

408. The fact that every point of the plane has one ellipse and one hyperbola of the confocal system (19) passing through it, enables us to regard the two values of the parameter λ that determine these two curves as co-ordinates of the point; they are called *elliptic co-ordinates.* If x, y be the rectangular cartesian co-ordinates of the point, its elliptic co-ordinates λ_1, λ_2 are found as the roots of the equation (19) which is quadratic in λ. Conversely, to transform from elliptic to cartesian co-ordinates, that is, to express x and y in terms of λ_1 and λ_2, we have only to solve for x and y the two equations

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1.$$

The two confocal conics that pass through the same point P intersect at right angles. For, the tangent to the ellipse at P bisects the exterior angle at P in the triangle SPS', while the tangent to the hyperbola bisects the interior angle at the same point; in other words, the tangent to one curve is normal to the other, and *vice versa.* The elliptic system of co-ordinates is, therefore, an *orthogonal* system; the infinitesimal elements $d\lambda_1 \cdot d\lambda_2$ into which the two series of confocal conics (19) divide the plane are rectangular, though curvilinear.

409. These considerations are easily extended to space of three dimensions. An ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ where } a>b>c,$$

has six real foci in its principal planes; two, S_1, S_1', in the xy-plane, on the axis of x, at a distance $OS_1 = \sqrt{a^2-b^2}$ from the center O; two, S_2, S_2', in the yz-plane, on the axis of y, at the distance $OS_2 = \sqrt{b^2-c^2}$ from the center; and two, S_3, S_3', in the zx-plane, on the axis of x, at the distance $OS_3 = \sqrt{a^2-c^2}$ from the center. It should be noticed that, since $b>c$, we have $OS_3 > OS_1$; *i. e.* S_1, S_1' lie between S_3, S_3' on the axis of x.

The same holds for hyperboloids.

Two quadric surfaces are said to be confocal when their principal sections are confocal conics. Now this will be the case for two quadric surfaces whose semi-axes are a_1, b_1, c_1, and a_2, b_2, c_2, if the directions of their axes coincide and if

$$a_1^2-b_1^2=a_2^2-b_2^2, \quad b_1^2-c_1^2=b_2^2-c_2^2, \quad a_1^2-c_1^2=a_2^2-c_2^2.$$

Writing these conditions in the form

$$a_2^2 - a_1^2 = b_2^2 - b_1^2 = c_2^2 - c_1^2, \quad \text{say} = \lambda,$$

we find $a_2^2 = a_1^2 + \lambda$, $b_2^2 = b_1^2 + \lambda$, $c_2^2 = c_1^2 + \lambda$. Hence the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \tag{20}$$

where λ is a variable parameter, represents a system of confocal quadric surfaces.

410. As long as λ is algebraically greater than $-c^2$, the equation (20) represents ellipsoids. For $\lambda = -c^2$ the surface collapses into the interior area of the ellipse in the xy-plane whose vertices are the foci S_2, S_2' and S_3, S_3'. For as λ approaches the limit $-c^2$, the three semi-axes of (20) approach the limits $\sqrt{a^2-c^2}$, $\sqrt{b^2-c^2}$, 0, respectively. This limiting ellipse is called the *focal ellipse*. Its foci are the points S_1, S_1', since $a^2 - c^2 - (b^2 - c^2) = a^2 - b^2$.

When λ is algebraically $< -c^2$, but $> -a^2$, the equation (20) represents hyperboloids; for values of $\lambda < -a^2$ it is not satisfied by any real points. As long as $-b^2 < \lambda < -c^2$, the surfaces are hyperboloids of one sheet. The limiting surface $\lambda = -c^2$ now represents the exterior area of the focal ellipse in the xy-plane. The limiting hyperboloid of one sheet for $\lambda = -b^2$ is the area in the zx-plane bounded by the hyperbola whose vertices are S_1, S_1', and whose foci are S_3, S_3'. This is called the *focal hyperbola*.

Finally, when $-a^2 < \lambda < -b^2$, the surfaces are hyperboloids of two sheets, the limiting hyperboloid $\lambda = -a^2$ collapsing into the yz-plane.

411. It appears from these geometrical considerations, that there are passing through every point of space three surfaces confocal to the fundamental ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and to each other, viz.: an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. This can also be shown analytically, as there is no difficulty in proving that the equation (20) has three real roots, say λ_1, λ_2, λ_3, for every set of real values of x, y, z, and that these roots are confined between such limits as to give the three surfaces just mentioned.

The quantities λ_1, λ_2, λ_3 can therefore be taken as co-ordinates of the point (x, y, z); and these *elliptic co-ordinates* of the point are, geometrically, the parameters of the three quadric surfaces passing through the point and confocal to the fundamental ellipsoid; while, analytically,

they are the three roots of the cubic (20). To express x, y, z in terms of the elliptic co-ordinates, it is only necessary to solve for x, y, z the three equations obtained by substituting in (20) successively λ_1, λ_2, λ_3 for λ.

412. The geometrical meaning of the parameter λ will appear by considering two parallel tangent planes π_0 and π_λ (on the same side of the origin), the former (π_0) tangent to the fundamental ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the latter ($\pi_\lambda$) tangent to any confocal surface λ or $x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) + z^2/(c^2 + \lambda) = 1$. The perpendiculars q_0, q_λ, let fall from the origin O on these tangent planes π_0, π_λ, are given by the relations (the proof being the same as in Art. 399)

$$q_0^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2, \tag{21}$$

$$q^{\lambda 2} = (a^2 + \lambda)\alpha^2 + (b^2 + \lambda)\beta^2 + (c^2 + \lambda)\gamma^2, \tag{22}$$

where α, β, γ are the direction cosines of the common normal of the planes π_0, π_λ. Subtracting (21) from (22), we find, since $\alpha^2 + \beta^2 + \gamma^2 = 1$,

$$q_\lambda^2 - q_0^2 = \lambda; \tag{23}$$

i. e. the parameter λ of any one of the confocal surfaces (20) *is equal to the difference of the squares of the perpendiculars let fall from the common center on any tangent plane to the surface λ, and on the parallel tangent plane to the fundamental ellipsoid* $\lambda = 0$.

413. Let us now apply these results to the question of the distribution of the principal axes throughout space.

We take the centroid G of the given body as origin, and select as fundamental ellipsoid of our confocal system the polar reciprocal of the central ellipsoid, *i. e.* the ellipsoid (17) formed for the centroid, for which the name "fundamental ellipsoid of the body" was introduced in Art. 401. Its equation is

$$\frac{x^2}{q_1^2} + \frac{y^2}{q_2^2} + \frac{z^2}{q_3^2} = 1,$$

if q_1, q_2, q_3 are the principal radii of inertia of the body.

The radius of inertia q_0 for any centroidal line l_0 can be constructed (Art. 400) by laying a tangent plane to this ellipsoid perpendicular to the line l_0; if this line meets the tangent plane at Q_0 (Fig. 85), then

$q_0 = GQ_0$. Analytically, if α, β, γ be the direction cosines of l_0, q_0 is given by formula (21) or (12′).

To find the radius of inertia q for a line l, parallel to l_0, and passing through any point P, we lay through P a plane π_λ, perpendicular to l, and a parallel plane π_0, tangent to the fundamental ellipsoid; let Q_λ,

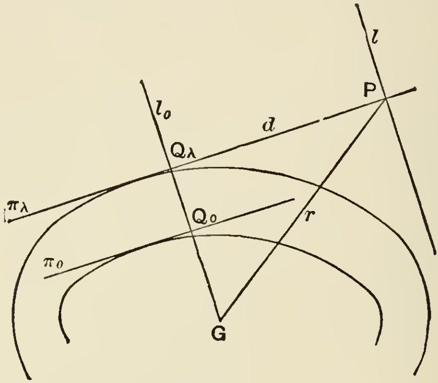

Fig. 85.

Q_0 be the intersections of these planes with the centroidal line l_0. Then, putting $GQ_0 = q_0$, $GQ_\lambda = q_\lambda$, $GP = r$, $PQ_\lambda = d$, we have, by Art. 385,

$$q^2 = q_0^2 + d^2.$$

The figure gives the relation $d^2 = r^2 - q_\lambda^2$, which, in combination with (23) reduces the expression for the radius of inertia for the line l to the simple form

$$q^2 = r^2 - \lambda. \tag{24}$$

414. The value of $r^2 - \lambda$, and hence the value of q, remains the same for the perpendiculars to all planes through P, tangent to the same quadric surface λ: these perpendiculars form, therefore, an equimomental cone at P. By varying λ we thus obtain all the equimomental cones at P. The principal diameters of all these cones coincide in direction, since they coincide with the directions of the principal axes of the momental ellipsoid at P (see Art. 396); but they also coincide with the principal diameters of the cones enveloped by the tangent planes π_λ. It thus appears that *the principal axes at the point P coincide in direction*

with the principal diameters of the tangent cone from P as vertex to the fundamental ellipsoid $x^2/q_1^2 + y^2/q_2^2 + z^2/q_3^2 = 1$.

Instead of the fundamental ellipsoid, we might have used any quadric surface λ confocal to it. In particular, we may select the confocal surfaces $\lambda_1, \lambda_2, \lambda_3$ that pa s through P. For each of these the cone of the tangent planes collapses into a plane, viz. the tangent plane to the surface at P, while the cone of the perpendiculars reduces to a single line, viz. the normal to the surface at P. Thus we find that *the principal axes at any point P coincide in direction with the normals to the three quadric surfaces, confocal to the fundamental ellipsoid and passing through P.*

For the magnitudes of the principal radii q_x, q_y, q_z, at P, we evidently have

$$q_x^2 = r^2 - \lambda_1, \quad q_y^2 = r^2 - \lambda_2, \quad q_z^2 = r^2 - \lambda_3.$$

CHAPTER XVII.

RIGID BODY WITH A FIXED AXIS.

415. A rigid body with a fixed axis has but one degree of freedom. Its motion is fully determined by the motion of any one of its points (not situated on the axis), and any such point must move in a circle about the axis. Any particular position of the body is, therefore, determined by a single variable, or co-ordinate, such as the angle of rotation. Just as the equilibrium of such a body depends on a single condition (see Art. 234), so its motion is given by a single equation.

This equation is obtained at once by "taking moments about the fixed axis." For, according to the proposition of angular momentum (Art. 360), the time-rate of change of angular momentum about any axis is equal to the moment of the external forces about this axis. Hence, denoting this moment by H and taking the fixed axis as axis of z, we have as equation of motion the last of the equations (3′), Art. 360, viz.,

$$\frac{d}{dt}\Sigma m(x\dot{y} - y\dot{x}) = H. \tag{1}$$

416. The *angular momentum*, $\Sigma m(x\dot{y} - y\dot{x})$, about the fixed axis can be reduced to a more simple form. For rotation of angular velocity ω about the z-axis we have (Art. 48, Ex. 1) $\dot{x} = -\omega y$, $\dot{y} = \omega x$, so that

$$\Sigma m(x\dot{y} - y\dot{x}) = \omega\Sigma m(x^2 + y^2) = \omega\cdot\Sigma mr^2 = I\omega.$$

where r is the distance of the particle m from the axis and $I = \Sigma mr^2$ the moment of inertia of the body for this axis.

This expression for the angular momentum can be derived without reference to any co-ordinate system. For evidently $m\omega r$ is the linear momentum of the particle m, $m\omega r^2$ is its moment, *i. e.* the angular momentum of the particle, about the axis; and $\Sigma m\omega r^2 = \omega\Sigma mr^2 = I\omega$ is the angular momentum of the body about the axis.

It thus appears that, just as in translation the linear momentum of a body is the product of its mass into its linear velocity, so in the case of rotation *the angular momentum of the body about the axis of rotation is the product of its moment of inertia* (for this axis) *into the angular velocity.*

As regards the right-hand member of equation (1), the reactions of the axis need not be taken into account in forming the moment H; for as these reactions meet the axis, their moments about this axis are zero.

417. Substituting $I\omega$ for $\Sigma m(x\dot{y} - y\dot{x})$ in equation (1), and observing that the moment of inertia I about a *fixed* axis remains constant, we find the **equation of motion** in the form

$$I\frac{d\omega}{dt} = H; \tag{2}$$

i. e. for rotation about a fixed axis *the product of the moment of inertia for this axis into the angular acceleration equals the moment of the external forces about this axis;* just as, in the case of rectilinear translation, the product of the mass of the body into the linear acceleration equals the resultant force R along the line of motion:

$$m\frac{dv}{dt} = R.$$

And just as the latter equation may serve to determine

experimentally the mass of a body by observing the acceleration produced in it by a given force R, *e. g.* the force of gravity (as in the gravitation system, Art. 177), so the former equation, (2), may serve to determine experimentally the moment of inertia of a body about a line l, by observing the angular acceleration produced in the body when rotating about l under given forces.

418. For the **kinetic energy** of a body rotating with angular velocity ω about any axis we have

$$T = \Sigma\tfrac{1}{2}mv^2 = \Sigma\tfrac{1}{2}m\omega^2r^2 = \tfrac{1}{2}I\omega^2, \tag{3}$$

an expression which is again similar in form to that for the kinetic energy of a body in translation, viz. $T = \frac{1}{2}mv^2$.

When the axis is fixed so that I is constant, the equation of motion (2), multiplied by ω and integrated, say from $t = 0$ to $t = t$, gives the relation

$$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0{}^2 = \int_0^t H\omega dt, \tag{4}$$

which expresses the *principle of kinetic energy and work.*

419. As an example consider the **compound pendulum**, *i. e.* a rigid body with a fixed *horizontal* axis and subject to gravity alone. If $OG = h$ is the distance of the centroid G from the fixed axis and θ the angle made by OG with the vertical plane through the axis we have $H = Mgh \sin\theta$. Denoting by q the radius of inertia about the centroidal axis through G parallel to the fixed axis so that the moment of inertia about the fixed axis is $= M(q^2 + h^2)$, we find the equation of motion (2) in the form

$$\ddot{\theta} = -\frac{gh}{q^2 + h^2}\sin\theta. \tag{5}$$

Comparing this with the equation of motion of the *simple*

pendulum (Arts. 63, 335), $\ddot{\theta} = -(g/l)\sin\theta$, it appears that *the motion of a compound pendulum is the same as that of a simple pendulum of length*

$$l = h + \frac{q^2}{h}. \tag{6}$$

This is called *the length of the equivalent simple pendulum.* The foot O of the perpendicular let fall from the centroid G on the fixed axis is called the *center of suspension.* If on the line OG a length $OC = l$ be laid off, the point C is called the *center of oscillation.* It appears, from (6), that G lies between O and C.

The relation (6) can be written in the form

$$h(l - h) = q^2, \text{ or } OG \cdot GC = \text{const.}$$

As this relation is not altered by interchanging O and C, it follows that *the centers of oscillation and suspension are interchangeable*; *i. e.* the period of a compound pendulum remains the same if it be made to swing about a parallel axis through the center of oscillation.

420. Exercises.

(1) A pendulum, formed of a cylindrical rod of radius a and length L, swings about a diameter of one of the bases. Find the time of a small oscillation.

(2) A cube, whose edge is a, swings as a pendulum about an edge. Find the length of the equivalent simple pendulum.

(3) A circular disk of radius r revolves uniformly about its axis, making 100 rev./min. What is its kinetic energy?

(4) A homogeneous straight rod of length l is hinged at one end so as to turn freely in a vertical plane. If it be dropped from a horizontal position, with what angular velocity does it pass through the vertical position? (Equate the kinetic energy to the work of gravity.)

(5) A homogeneous plate whose shape is that of the segment of a parabola bounded by the curve and its latus rectum swings about the

latus rectum which is horizontal. Find the length of the equivalent simple pendulum.

(6) When q is given while l and h vary, the equation (6) represents a hyperbola whose asymptotes are the axis of l and the bisector of the angle between the (positive) axes of h and l. Show that $l_{\min} = 2q$ for $h = q$; also that l, and hence the period of oscillation, can be made very large by taking h either very large or very small. The latter case occurs for a ship whose *metacenter* (which plays the part of the point of suspension) lies very near its centroid.

(7) A homogeneous circular disk, 1 ft. in diameter and weighing 25 lbs., is making 240 rev./min. when left to itself. Determine the constant tangential force applied to its rim that would bring it to rest in 1 min.

421. While a single equation determines the motion of a body with a fixed axis, the other five equations of motion of a rigid body must be used to determine the **reactions.**

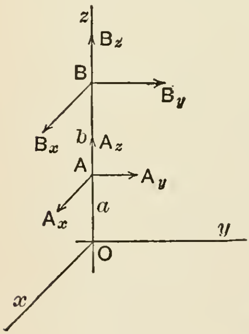

Fig. 86.

The axis will be fixed if any two of its points A, B are fixed. The reaction of the fixed point A can be resolved into three components A_x, A_y, A_z, that of B into B_x, B_y, B_z. By introducing these reactions the body becomes free; and the system composed of these reactions, of the external forces, and of the reversed effective forces must be in equilibrium. We take the axis of rotation as axis of z (Fig. 86) so that the z-co-ordinates of the particles are constant, and hence $\dot{z} = 0$, $\ddot{z} = 0$; and we put $OA = a$, $OB = b$. Then the six equations of motion are (see Art. 359 (2) and Art. 360 (3)):

$$\Sigma m\ddot{x} = \Sigma X + A_x + B_x,$$
$$\Sigma m\ddot{y} = \Sigma Y + A_y + B_y,$$

$$
\begin{aligned}
0 &= \Sigma Z + A_z + B_z, \\
-\Sigma mz\ddot{y} &= \Sigma(yZ - zY) - aA_y - bB_y, \\
\Sigma mz\ddot{x} &= \Sigma(zX - xZ) + aA_x + bB_x, \\
\Sigma m(x\ddot{y} - y\ddot{x}) &= \Sigma(xY - yX).
\end{aligned}
$$

422. It remains to introduce into these equations the values for $\ddot{x}$, $\ddot{y}$. As the motion is a pure rotation, we have (see Art. 48, Ex. 1) $\dot{x} = -\omega y$, $\dot{y} = \omega x$; hence, $\ddot{x} = -\dot{\omega} y - \omega^2 x$, $\ddot{y} = \dot{\omega} x - \omega^2 y$. Summing over the whole body, we find

$$
\begin{aligned}
\Sigma m\ddot{x} &= -\dot{\omega}\Sigma my - \omega^2\Sigma mx = -M\dot{\omega}\bar{y} - M\omega^2\bar{x}, \\
\Sigma m\ddot{y} &= \quad \dot{\omega}\Sigma mx - \omega^2\Sigma my = \quad M\dot{\omega}\bar{x} - M\omega^2\bar{y},
\end{aligned}
$$

where $\bar{x}$, $\bar{y}$ are the co-ordinates of the centroid; and

$$
\begin{aligned}
-\Sigma mz\ddot{y} &= -\dot{\omega}\Sigma mzx + \omega^2\Sigma myz = -E\dot{\omega} + D\omega^2, \\
\Sigma mz\ddot{x} &= -\dot{\omega}\Sigma myz - \omega^2\Sigma mzx = -D\dot{\omega} - E\omega^2,
\end{aligned}
$$

$$\Sigma m(x\ddot{y} - y\ddot{x}) = \dot{\omega}\Sigma mx^2 - \omega^2\Sigma mxy + \dot{\omega}\Sigma my^2 + \omega^2\Sigma mxy = C\dot{\omega},$$

where $C = \Sigma m(x^2 + y^2)$, $D = \Sigma myz$, $E = \Sigma mzx$ are the notations introduced in Art. 387.

With these values the equations of motion assume the form:

$$
\begin{aligned}
-M\bar{x}\omega^2 - M\bar{y}\dot{\omega} &= \Sigma X + A_x + B_x, \\
-M\bar{y}\omega^2 + M\bar{x}\dot{\omega} &= \Sigma Y + A_y + B_y, \\
0 &= \Sigma Z + A_z + B_z, \\
D\omega^2 - E\dot{\omega} &= \Sigma(yZ - zY) - aA_y - bB_y, \\
-E\omega^2 - D\dot{\omega} &= \Sigma(zX - xZ) + aA_x + bB_x, \\
C\dot{\omega} &= \Sigma(xY - yX).
\end{aligned}
\tag{7}
$$

423. The last equation is identical with equation (2), Art. 417.

The components of the reactions along the axis of rotation occur only in the third equation and can therefore not be found separately. The longitudinal pressure on the axis is

$$= -A_z - B_z = \Sigma Z.$$

The remaining four equations are sufficient to determine A_x, A_y, B_x, B_y.

The total stress to which the axis is subject, instead of being represented by the two forces, at A and B, can be reduced for the origin O to a force and a couple. The equations (7) give for the components of the force

$$\begin{aligned} -A_x - B_x &= \Sigma X + M\bar{x}\omega^2 + M\bar{y}\dot{\omega}, \\ -A_y - B_y &= \Sigma Y + M\bar{y}\omega^2 - M\bar{x}\dot{\omega}, \\ -A_z - B_z &= \Sigma Z. \end{aligned} \tag{8}$$

This force consists of the resultant of the external forces,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

and two forces in the xy-plane which form the reversed effective force of the centroid; for $M\bar{x}\omega^2$ and $M\bar{y}\omega^2$ give as resultant the centrifugal force $M\omega^2\sqrt{\bar{x}^2 + \bar{y}^2} = M\omega^2\bar{r}$, directed from the origin towards the projection of the centroid on the xy-plane, while $M\bar{y}\dot{\omega}$, $-M\bar{x}\dot{\omega}$ form the tangential resultant $M\dot{\omega}\bar{r}$, perpendicular to the plane through axis and centroid.

The couple has a component in the yz-plane, and one in the zx-plane, viz.:

$$\begin{aligned} aA_y + bB_y &= \Sigma(yZ - zY) - D\omega^2 + E\dot{\omega}, \\ -aA_x - bB_x &= \Sigma(zX - xZ) + E\omega^2 + D\dot{\omega}, \end{aligned} \tag{9}$$

while the component in the xy-plane is zero. The resultant couple lies, therefore, in a plane passing through the axis of rotation.

424. In the particular case *when no forces X, Y, Z are acting on the body*, the last of the equations (7), or equation (2), shows that *the angular velocity ω remains constant.* The stress on the axis of rotation will, however, exist; and the axis will in general tend to change both its direction, owing to the couple (9), and its position, owing to the force (8).

If the axis be not fixed as a whole, but only one of its points, the origin, be fixed, the force (8) is taken up by the fixed point, while the couple (9) will change the direction of the axis. Now this couple vanishes if, in addition to the absence of external forces, the conditions

$$D \equiv \Sigma myz = 0, \quad E \equiv \Sigma mzx = 0 \tag{10}$$

are fulfilled. In this case the body would continue to rotate about the axis of z even if this axis were not fixed, provided that the origin is a fixed point. A line having this property is called a **permanent axis of rotation.**

As the meaning of the conditions (10) is that the axis of z is a principal axis of inertia at the origin (see Art. 395), we have the proposition that *if a rigid body with a fixed point, not acted upon by any forces, begin to rotate about one of the principal axes at this point, it will continue to rotate uniformly about the same axis.* In other words the principal axes at any point are always, and are the only, permanent axes of rotation. This can be regarded as the dynamical definition of principal axes.

425. It appears from the equations (8) that the *position* of the axis of rotation will remain the same if, in addition to the absence of external forces, the conditions

$$\bar{x} = 0, \quad \bar{y} = 0 \tag{11}$$

be fulfilled; for in this case the components of the force (8) all vanish. If, moreover, the axis of rotation be a principal axis, the rotation will continue to take place about the same line even when the body has no fixed point.

The conditions (11) mean that the centroid lies on the axis of z; and it is known (Art. 395) that a centroidal principal axis is a principal axis at every one of its points. The axis

of z must therefore be a principal axis of the body, *i. e.* a principal axis at the centroid. We have thus the proposition: *If a free rigid body, not acted upon by any forces, begin to rotate about one of its centroidal principal axes, it will continue to rotate uniformly about the same line.*

CHAPTER XVIII.

RIGID BODY WITH A FIXED POINT.

1. The general equations of motion.

426. If the fixed point O be taken as origin and the reaction at O be denoted by A (as in Art. 233) the equations of motion (2), (3) of Arts. 359, 360 become:

$$\Sigma m\ddot{x} = \Sigma X + A_x, \quad \Sigma m\ddot{y} = \Sigma Y + A_y, \quad \Sigma m\ddot{z} = \Sigma Z + A_z, \quad (1)$$

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY), \quad \Sigma m(z\ddot{x} - x\ddot{z}) = \Sigma(zX - xZ),$$
$$\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX). \quad (2)$$

The equations (1) merely serve to determine the reaction A, while the equations (2) determine the motion. There should be three such equations because a rigid body with a fixed point has three degrees of freedom (Art. 233)

Kinematically, the instantaneous state of motion is a rotation about an axis through O and is given by the rotor ω (Arts. 116, 128). The course of the motion consists of the rolling of the cone of body axes over the cone of space axes (Art. 131).

Dynamically, the instantaneous state of motion of the body is given by the *impulse-vector* h (Art. 367) which is the resultant of the angular momenta of all the particles constituting the body, or (Arts. 372, 373) the vector of that impulsive couple which, acting on the body at rest, would impart to it its instantaneous state of motion, *i. e.* would produce instantaneously the rotor ω. The given external forces reduce to a resultant R through O, which is taken

up by the fixed point and does not affect the motion, and a couple, of vector H, whose components are the right-hand members of (2). Writing these equations in the form (3″) of Art. 361, viz.

$$\frac{dh_x}{dt} = H_x, \quad \frac{dh_y}{dt} = H_y, \quad \frac{dh_z}{dt} = H_z, \tag{2'}$$

we see that the time-rate of change of the vector h is geometrically equal to the vector H.

The main question is the relation between the vectors ω and h.

427. Now for the angular momentum about the axis Ox we have since $\dot{x} = \omega_y z - \omega_z y$, $\dot{y} = \omega_z x - \omega_x z$, $\dot{z} = \omega_x y - \omega_y x$ (Art. 118):

$$h_x = \Sigma m(y\dot{z} - z\dot{y}) = \omega_x \Sigma m(y^2 + z^2) - \omega_y \Sigma mxy - \omega_z \Sigma mzx,$$

or, with the notation of Art. 387, $h_x = A\omega_x - F\omega_y - E\omega_z$. Determining h_y, h_z in the same way we find:

$$\begin{aligned} h_x &= \quad A\omega_x - F\omega_y - E\omega_z, \\ h_y &= - F\omega_x + B\omega_y - D\omega_z, \\ h_z &= - E\omega_x - D\omega_y + C\omega_z. \end{aligned} \tag{3}$$

These equations enable us to find the vector h when ω is given, and vice versa. The relation between these vectors which are evidently in general not parallel appears from the equation of the momental ellipsoid, (10), Art. 392. If we select the arbitrary constant ϵ so that this ellipsoid passes through the extremity of the rotor ω, that is so that

$$A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y = M\epsilon^4,$$

it appears that h_x, h_y, h_z are one half the partial derivatives of the left-hand member of this equation, and hence

the vector h is normal to the tangent plane to the momental ellipsoid at the extremity of ω; in other words, *the plane of the impulsive couple h is conjugate to the direction of ω with respect to the momental ellipsoid.*

428. For the kinetic energy we have if r is the distance of the particle m from the instantaneous axis ω:

$$T = \Sigma\tfrac{1}{2}mv^2 = \tfrac{1}{2}I\omega^2, \tag{4}$$

where $I = \Sigma mr^2$ is the moment of inertia of the body about the instantaneous axis. Now if α, β, γ are the direction cosines of this axis, *i. e.* of the rotor ω, we have by (9), Art. 390,

$$I = A\alpha^2 + B\beta^2 + C\gamma^2 - 2D\beta\gamma - 2E\gamma\alpha - 2F\alpha\beta;$$

multiplying by $\frac{1}{2}\omega^2$ we find

$$T = \tfrac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y). \tag{5}$$

It follows by (3) that

$$h_x = \frac{\partial T}{\partial \omega_x}, \quad h_y = \frac{\partial T}{\partial \omega_y}, \quad h_z = \frac{\partial T}{\partial \omega_z}. \tag{6}$$

Multiplying (3) by ω_x, ω_y, ω_z and adding we find

$$h_x\omega_x + h_y\omega_y + h_z\omega_z = 2T, \tag{7}$$

which means that *the kinetic energy is one half the dot-product $h \cdot \omega$ of the vectors h and ω.*

429. All these relations become far more simple if we take as axes of co-ordinates the principal axes at O; but it must be kept in mind that these are *moving* axes. Distinguishing, as in Kinematics, components along moving axes by the subscripts 1, 2, 3 instead of x, y, z, and denoting the principal moments of inertia at O by I_1, I_2, I_3 (Art. 393) we have by (3)

$$h_1 = I_1\omega_1, \quad h_2 = I_2\omega_2, \quad h_3 = I_3\omega_3. \tag{8}$$

These equations show that if the vector of an impulsive couple is parallel to a principal axis at O, it produces an angular velocity about this axis; it follows from the equations (3) that the condition is not only sufficient but necessary. Comp. Art. 424.

For the kinetic energy we have by (12), Art. 393:

$$T = \Sigma \tfrac{1}{2}mv^2 = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\omega^2(I_1\alpha^2 + I_2\beta^2 + I_3\gamma^2)$$
$$= \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Substituting for I_1, I_2, I_3, or for ω_1, ω_2, ω_3 their values from (8) we find

$$\begin{aligned} 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\ &= h_1\omega_1 + h_2\omega_2 + h_3\omega_3 \qquad (9) \\ &= \frac{h_1^2}{I_1} + \frac{h_2^2}{I_2} + \frac{h_3^2}{I_3}. \end{aligned}$$

430. Euler's Equations. It appears from the equations (2′) that the impulse h which, by (3) or (8), determines ω and hence the instantaneous state of motion of the body, varies in the course of the motion, under the action of the external forces both in magnitude and in direction, and also both relatively to the body and relatively to the fixed trihedral of axes.

It is generally found most convenient to determine first the variation of the vector h relatively to the moving axes, and then to determine the motion of the trihedral of the moving axes with respect to the fixed axes. The former of these problems is solved by Euler's equations (Art. 432) while the latter can be solved with the aid of Euler's angles (Art. 434) or any other suitable parameters.

431. Euler's equations are essentially the equations (2′)

when referred to the principal axes at O; they express the geometrical relation $dh/dt = H$.

The variation of the vector h (drawn from O) depends on the motion of its extremity whose co-ordinates are h_x, h_y, h_z with respect to the fixed axes, and h_1, h_2, h_3 with respect to the moving axes (for the present, not necessarily the principal axes). The absolute velocity $(\dot{h}_x, \dot{h}_y, \dot{h}_z)$ of the extremity of h can be resolved into its relative velocity $(\dot{h}_1, \dot{h}_2, \dot{h}_3)$ and the body-velocity $\omega \times h$ (Arts. 118, 119, 129) whose components along the moving axes are $\omega_2 h_3 - \omega_3 h_2$, $\omega_3 h_1 - \omega_1 h_3$, $\omega_1 h_2 - \omega_2 h_1$. The equations (2′) referred to any moving axes fixed in the body become therefore

$$\begin{aligned} \frac{dh_1}{dt} + \omega_2 h_3 - \omega_3 h_2 &= H_1, \\ \frac{dh_2}{dt} + \omega_3 h_1 - \omega_1 h_3 &= H_2, \\ \frac{dh_3}{dt} + \omega_1 h_2 - \omega_2 h_1 &= H_3; \end{aligned} \tag{10}$$

or briefly, in vector form: $\dot{h} + \omega \times h = H$

432. If, in particular, we take as moving axes the principal axes at O, the equations (10), owing to the relations (8), reduce to the following:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = H_1, \quad I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = H_2,$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = H_3, \tag{11}$$

which are known as **Euler's equations** of motion of a rigid body with a fixed point. Their integration gives ω_1, ω_2, ω_3, and hence ω, as functions of the time t.

433. Analytically, the equations (10) can be derived from the equations (2), Art. 426, or rather from the corresponding equations for fixed axes coinciding at the instant considered with the moving axes, viz.:

$\Sigma m(y_1\ddot{z}_1 - z_1\ddot{y}_1) = H_1, \quad \Sigma m(z_1\ddot{x}_1 - x_1\ddot{z}_1) = H_2, \quad \Sigma m\,(x_1\ddot{y}_1 - y_1\ddot{x}_1) = H_3,$

by introducing for $\ddot{x}_1$, $\ddot{y}_1$, $\ddot{z}_1$ their values from (4′), Art. 141.

We thus find for $\Sigma m(y_1\ddot{z}_1 - z_1\ddot{y}_1)$:

$$\begin{aligned}&\omega_3(\omega_1\Sigma mx_1y_1 + \omega_2\Sigma my^2 + \omega_3\Sigma my_1z_1) - \omega^2\Sigma my_1z_1\\ &- \omega_2(\omega_1\Sigma mz_1x_1 + \omega_2\Sigma my_1z_1 + \omega_3\Sigma mz_1^2) + \omega^2\Sigma my_1z_1\\ &+ \dot{\omega}_1\Sigma m(y_1^2 + z_1^2) - \dot{\omega}_2\Sigma mx_1y_1 - \dot{\omega}_3\Sigma mz_1x_1,\end{aligned}$$

or with the notation of Art. 387:

$$\begin{aligned}&\omega_3(F\omega_1 + C\omega_2 + D\omega_3) - \omega_2(E\omega_1 + D\omega_2 + B\omega_3) + A\dot{\omega}_1 - F\dot{\omega}_2 - E\dot{\omega}_3\\ &= \omega_2(-E\omega_1 - D\omega_2 + C\omega_3) - \omega_3(-F\omega_1 + B\omega_2 - D\omega_3) + A\dot{\omega}_1 - F\dot{\omega}_2 - E\dot{\omega}_3\\ &= \omega_2h_3 - \omega_3h_2 + \frac{dh_1}{dt},\end{aligned}$$

by (3), Art. 427. The relations (3) hold of course for moving axes as well as for fixed axes. But for the fixed axes the coefficients of ω_x, ω_y, ω_z, *i. e.* the moments and products of inertia for the fixed axes, are not constant, while for the moving axes the coefficients of ω_1, ω_2, ω_3 are constant.

434. The position of the moving trihedral at any instant with respect to the fixed trihedral can be assigned by three angles as follows. Let X, Y, Z (Fig. 87) be the intersections of the fixed axes, X_1, Y_1, Z_1 those of the moving axes with the sphere of radius **1** described about the fixed point O; and let N be the intersection with the same sphere of the *nodal line*, or *line of nodes*, *i. e.* the line in which the planes XOY and X_1OY_1 intersect. Then the angles $ZZ_1 = \theta$, $NX_1 = \varphi$, $XN = \psi$, usually called **Euler's angles**, fully

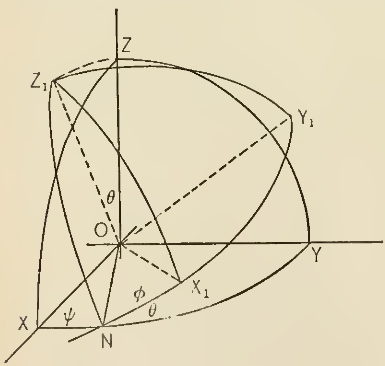

Fig. 87.

determine the relative position of one trihedral with respect to the other. If the moving trihedral be initially coincident with the fixed trihedral it can be carried into any other position in three steps: (*a*) turn the trihedral $X_1Y_1Z_1$, when coincident with XYZ, about OZ counterclockwise until OX_1 coincides with the assumed positive sense of the nodal line ON, and call the angle of this rotation ψ; (*b*) in the new position turn $X_1Y_1Z_1$ counterclockwise about ON until the plane X_1OY_1 falls into its final position, the angle of this rotation is θ; (*c*) finally turn $X_1Y_1Z_1$ about OZ_1 counterclockwise through an angle φ until OX_1 reaches its final position.

435. The rotor ω can evidently be resolved along the axes ON, OZ_1, OZ into the components $\dot\theta$, $\dot\varphi$, $\dot\psi$; hence the sum of the projections of these components $\dot\theta$, $\dot\varphi$, $\dot\psi$ on OX_1 must be equal to ω_1; similarly for ω_2, ω_3. As Fig. 87 shows, the direction cosines of ON, OZ_1, OZ with respect to the moving trihedral are

	X_1	Y_1	Z_1
N	$\cos\varphi$	$-\sin\varphi$	0
Z_1	0	0	1
Z	$\sin\theta\,\sin\varphi$	$\sin\theta\,\cos\varphi$	$\cos\theta$

Hence

$$\begin{aligned}\omega_1 &= \dot\theta\,\cos\varphi + \dot\psi\,\sin\theta\,\sin\varphi,\\ \omega_2 &= -\,\dot\theta\,\sin\varphi + \dot\psi\,\sin\theta\,\cos\varphi,\\ \omega_3 &= \dot\varphi + \dot\psi\,\cos\theta.\end{aligned} \tag{12}$$

By substituting these values in Euler's equations we obtain differential equations of the second order for θ, φ, ψ. If Euler's equations have been solved so that ω_1, ω_2, ω_3 have been found as functions of t, the equations (12) are differential equations of the first order for θ, φ, ψ. Solving these equations for $\dot\theta$, $\dot\varphi$, $\dot\psi$ we have:

$$\begin{aligned}\dot{\theta} &= \omega_1 \cos\varphi - \omega_2 \sin\varphi,\\ \dot{\varphi} &= -\omega_1 \sin\varphi \cot\theta - \omega_2 \cos\varphi \cot\theta + \omega_3,\\ \dot{\psi} &= \omega_1 \sin\varphi \csc\theta + \omega_2 \cos\varphi \csc\theta.\end{aligned} \tag{12'}$$

2. Motion without forces.

436. Let a rigid body with a fixed point O be given an initial angular velocity about an axis through O, and let the resultant couple H of the external forces be zero. By Art. 427, the initial position of the body, *i. e.* of its momental ellipsoid, together with the initial axis of rotation, determines the initial direction of the impulse h, this direction being perpendicular to the tangent plane to the ellipsoid at the point P where it is met by the instantaneous axis (Fig. 88).

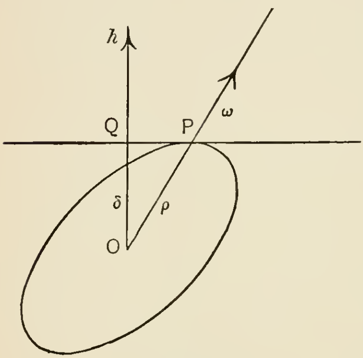

Fig. 88.

As H is zero, it follows from (2′), Art. 426, that h is constant in magnitude and direction. Moreover, by (9), Art. 429, the kinetic energy T is constant. Finally, it can be shown that the perpendicular δ let fall from O on the tangent plane at P is constant.

To prove this let

$$I_1 x_1^2 + I_2 y_1^2 + I_3 z_1^2 = 1$$

be the equation of the momental ellipsoid referred to the principal axes so that the tangent plane at P (ξ, η, ζ) has the equation

$$I_1 x_1 \xi + I_2 y_1 \eta + I_3 z_1 \zeta = 1.$$

If ρ be the radius vector OP of P we have

$$\frac{\xi}{\omega_1} = \frac{\eta}{\omega_2} = \frac{\zeta}{\omega_3} = \frac{\rho}{\omega}.$$

Hence

$$\frac{1}{\delta} = \sqrt{I_1^2\xi^2 + I_2^2\eta^2 + I_3^2\zeta^2} = \frac{\rho}{\omega}\sqrt{I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2}$$
$$= \frac{\rho}{\omega}h,$$

by (8), Art. 429. On the other hand, as P lies on the ellipsoid we have

$$I_1\xi^2 + I_2\eta^2 + I_3\zeta^2 = 1, \quad i.\ e.\ \ \frac{\rho^2}{\omega^2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = 1.$$

By (9), Art. 429, this shows that $\rho/\omega = 1/\sqrt{2T}$. Hence

$$\frac{1}{\delta} = \frac{h}{\sqrt{2T}},$$

and as both h and T are constant, δ is constant.

From the relation between the directions of ω and h and the constancy of h and δ it follows that *the motion of the body consists in the rolling of its momental ellipsoid over a fixed tangent plane.*

437. The points where the instantaneous axis meets the momental ellipsoid form a curve, fixed in the body and moving with it, which is called the **polhode** (path of the pole P). The intersections of the instantaneous axis with the fixed tangent plane form another curve, called the **herpolhode,** which is fixed in space. The cones projecting these curves from O are known as *Poinsot's rolling cones*, the polhodal cone rolling over the fixed herpolhodal cone.

438. The equations of the *polhode* as the locus of those points of the momental ellipsoid whose tangent plane has the

constant distance δ from O are evidently

$$I_1x_1^2 + I_2y_1^2 + I_3z_1^2 = 1, \quad I_1^2x_1^2 + I_2^2y_1^2 + I_3^2z_1^2 = \frac{1}{\delta^2};$$

i. e. the polhode is the intersection of the momental ellipsoid with a coaxial ellipsoid. Multiplying the second equation by δ^2 and subtracting the result from the first equation we obtain the equation of the *polhodal cone*

$$I_1(1 - I_1\delta^2)x_1^2 + I_2(1 - I_2\delta^2)y_1^2 + I_3(1 - I_3\delta^2)z_1^2 = 0.$$

If we take the notation so that $I_1 > I_2 > I_3$ this cone is real if and only if

$$\frac{1}{I_1} < \delta^2 < \frac{1}{I_3}.$$

For $\delta^2 = 1/I_3$, the polhode reduces to a point, viz. the extremity of the longest axis of the momental ellipsoid. As δ^2 diminishes, the polhode is first an oval about this longest axis. When $\delta^2 = 1/I_2$, the polhodal cone degenerates into a pair of planes and the polhode becomes an ellipse. When δ^2 lies between $1/I_2$ and $1/I_1$ the polhode is an oval about the shortest axis, and it contracts to the extremity of this axis for $\delta^2 = 1/I_1$.

For values of δ^2 very close to $1/I_2$ the motion can, in a certain sense, be called unstable since a slight disturbance might change the polhodal cone from a cone about the longest to a cone about the shortest axis, or vice versa.

439. The *herpolhode* is a plane curve; but it is in general not closed. The radius vector $OP = \rho$ (Fig. 88), if not constant, has a greatest and a least value in the course of the motion, and the same is true of its projection QP on the fixed plane. Hence the herpolhode lies between two concentric circles. When ρ is constant these circles coincide

and the herpolhode coincides with them. It can be shown that the herpolhode has no points of inflection.

440. The invariable line describes a cone in the moving body. Its equation may be found from the reciprocal ellipsoid

$$\frac{x_1^2}{I_1} + \frac{y_1^2}{I_2} + \frac{z_1^2}{I_3} = 1,$$

whose radius vector in the direction δ is $1/\delta$ (Arts. 398, 399), and hence constant. The cone must pass through the intersection of the reciprocal ellipsoid and the sphere

$$x_1^2 + y_1^2 + z_1^2 = \frac{1}{\delta^2}.$$

Hence its equation is

$$\left(\delta^2 - \frac{1}{I_1}\right) x_1^2 + \left(\delta^2 - \frac{1}{I_2}\right) y_1^2 + \left(\delta^2 - \frac{1}{I_3}\right) z_1^2 = 0.$$

441. When $H = 0$ *Euler's equations* (11), Art. 432, are

$$\begin{gathered} I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \\ I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2. \end{gathered} \tag{13}$$

Multiplying by ω_1, ω_2, ω_3 and adding we find

$$\frac{d}{dt}\tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = 0;$$

hence, by (9), Art. 429,

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T = \text{const.} \tag{14}$$

This is the *integral of kinetic energy and work.*

Multiplying (13) by $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$ and adding we find similarly by (3):

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = h^2 = \text{const.}, \tag{15}$$

which is the *integral of angular momentum.*

As, moreover,

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2, \tag{16}$$

we can solve (14), (15), (16) for ω_1^2, ω_2^2, ω_3^2. Introducing the new constants α, β, γ by putting

$$2T(I_2 + I_3) - h^2 = I_2 I_3 \alpha^2, \quad 2T(I_3 + I_1) - h^2 = I_3 I_1 \beta^2,$$

$$2T(I_1 + I_2) - h^2 = I_1 I_2 \gamma^2,$$

we find

$$\begin{aligned} \omega_1^2 &= \frac{I_2 I_3}{(I_1 - I_2)(I_1 - I_3)} (\omega^2 - \alpha^2), \\ \omega_2^2 &= \frac{I_3 I_1}{(I_2 - I_3)(I_1 - I_2)} (\beta^2 - \omega^2), \\ \omega_3^2 &= \frac{I_1 I_2}{(I_1 - I_3)(I_2 - I_3)} (\omega^2 - \gamma^2). \end{aligned} \tag{17}$$

Hence, if $I_1 > I_2 > I_3$ we have $\omega^2 > \alpha^2$, $\omega^2 < \beta^2$, $\omega^2 > \gamma^2$.

442. To find the *time*, multiply the equations (13) by ω_1/I_1, ω_2/I_2, ω_3/I_3 and add:

$$d(\tfrac{1}{2}\omega^2) = \frac{(I_1 - I_2)(I_1 - I_3)(I_2 - I_3)}{I_1 I_2 I_3} \omega_1 \omega_2 \omega_3;$$

substituting for ω_1, ω_2, ω_3 their values (17) we find:

$$t = \pm \tfrac{1}{2} \int \frac{d(\omega^2)}{\sqrt{(\omega^2 - \alpha^2)(\beta^2 - \omega^2)(\omega^2 - \gamma^2)}}.$$

The positive or negative sign must be used according as $d(\omega^2)$ is positive or negative.

As t is given by an elliptic integral, ω^2 is a periodic function of the time.

443. If, in particular, *the momental ellipsoid at O is an ellipsoid of revolution*, say if $I_1 = I_2$, the results assume a very simple form. Euler's equations (13) reduce to

$$\dot{\omega}_1 = \lambda \omega_2 \omega_3, \quad \dot{\omega}_2 = - \lambda \omega_3 \omega_1, \quad \dot{\omega}_3 = 0, \tag{18}$$

where

$$\lambda = \frac{I_2 - I_3}{I_1} = -\frac{I_3 - I_1}{I_2}.$$

The angular velocity ω_3 about the third axis Oz_1 (which is not necessarily an axis of symmetry for the mass of the whole body) is therefore constant:

$$\omega_3 = n.$$

The first two equations (18) give $\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$, whence

$$\omega_1^2 + \omega_2^2 = \text{const.} = m^2.$$

It follows that

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{m^2 + n^2}$$

is constant although ω_1 and ω_2 vary.

The inclination of the instantaneous axis to the principal axes Ox_1, Oy_1 varies, but its inclination to the third principal axis Oz_1 is constant, viz. $\cos^{-1}(\omega_3/\omega)$. This means that the polhodal cone is a cone of revolution about Oz_1 and the polhode is a circle. The herpolhode is therefore likewise a circle (Art. 439). As the two circular cones are in contact along the instantaneous axis, this axis lies in the same plane with the impulse h and the axis Oz_1.

444. To find ω_1, ω_2 separately, differentiate the first equation (18) with respect to t and substitute for $\dot{\omega}_2$ its value from the second:

$$\ddot{\omega}_1 + \lambda^2 n^2 \omega_1 = 0;$$

hence

$$\omega_1 = k \sin(\lambda n t + \epsilon),$$

where k, ϵ are the constants of integration. The first equation (18) then gives

$$\omega_2 = \frac{1}{\lambda n}\dot{\omega}_1 = k \cos(\lambda n t + \epsilon).$$

As $\omega_1^2 + \omega_2^2 = m^2$ (Art. 443) it appears that $k = m$. Hence

$$\omega_1 = m \sin(\lambda nt + \epsilon), \quad \omega_2 = m \cos(\lambda nt + \epsilon), \quad \omega_3 = n. \quad (19)$$

445. To determine the position of the body with respect to fixed axes through O let the invariable direction of h be taken as axis Oz. The direction cosines of h given in Art. 435 give

$$h_1 = I_1\omega_1 = h \sin\theta \sin\varphi, \quad h_2 = I_2\omega_2 = h \sin\theta \cos\varphi,$$
$$h_3 = I_3\omega_3 = h \cos\theta.$$

It follows that

$$\cos\theta = \frac{I_3 n}{h} = \text{const.}, \quad \tan\varphi = \frac{\omega_1}{\omega_2} = \tan(\lambda nt + \epsilon);$$

hence $\varphi = \lambda nt + \epsilon$ and $\dot{\theta} = 0$, $\dot{\varphi} = \lambda n = \text{const.}$

Finally, the third of the equations (12), Art. 435, gives

$$\dot{\psi} = \frac{n - \lambda n}{\cos\theta} = \frac{(1 - \lambda)h}{I_3} = \frac{h}{I_1},$$

whence $\psi = (h/I_1)t + \psi_0$.

Thus if we resolve ω along the oblique axes ON, OZ_1, OZ (Art. 435) into $\dot{\theta}$, $\dot{\varphi}$, $\dot{\psi}$ (see Fig. 87), we have $\dot{\theta} = 0$ while $\dot{\varphi}$ and $\dot{\psi}$ are constant. The motion of the body consists therefore in the rotation of constant angular velocity $\dot{\varphi} = \lambda n$ about OZ_1, together with the turning of this axis OZ_1 with constant angular velocity $\dot{\psi} = h/I_1$ about the axis OZ, the angle $\theta = ZOZ_1$ between these axes remaining constant. Such a motion is called a **regular precession**; the nodal line ON (Fig. 87) is said to *precess* with the *velocity of precession* $\dot{\psi}$; OZ is the *axis of precession*.

If, in particular, the momental ellipsoid at O is a sphere, so that $I_1 = I_2$ and hence $\lambda = 0$, we have $\dot{\varphi} = 0$; hence the

whole motion consists of the rotation of angular velocity $\dot{\psi}$ about the fixed axis OZ. This was to be expected; for, as a principal axis, OZ is a permanent axis of rotation (Art. 424).

3. Heavy symmetric top.

446. A rigid body with a fixed point is often spoken of as a **top** although the ordinary children's top has no fixed point but has merely one of its points approximately confined to a plane or other surface.

If the momental ellipsoid at the fixed point O is an ellipsoid of revolution, say about Oz_1, so that $I_1 = I_2$, and the centroid G of the body lies on Oz_1, say at the distance $OG = k$ from O, the body is called a *symmetric top*. If, moreover, the only force acting on the body (besides the reaction at O) is the weight W of the body we have the *heavy symmetric top*.

If k were zero we should have the case of Arts. 443–445. If $k \neq 0$ but the initial angular velocity be zero, the body would swing like a compound pendulum in a vertical plane. With proper initial conditions the heavy symmetric top may move like a (compound) spherical pendulum with $I_1 = I_2$ at O. But in speaking of the motion of the heavy symmetric top it is generally understood that the initial angular velocity is large and takes place about an axis not differing very much from the axis Oz_1. To explain what is here meant by large observe that if in the course of the motion the centroid G rises or descends through a vertical distance z the work of gravity, $\pm Wz$, changes the kinetic energy of the top. Now this variation in the kinetic energy can never amount to more than $2Wk$. Hence if k is reasonably small and the initial angular velocity large, the initial kinetic energy will not be affected very much by the changes due to the rise and fall of the centroid G. It is especially cases of this kind that we

have in mind when speaking of the *phenomena of the top.* The general equations of Arts. 447, 448, however, do not imply any such restricting assumptions.

447. Taking the fixed axis Oz vertical and positive upward and the moving axis Oz_1 along the third principal axis at O, we find Euler's equations (11) in the form

$$I_1\dot{\omega}_1 + (I_3 - I_1)\omega_2\omega_3 = Wk \sin\theta \cos\varphi,$$
$$I_1\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = - Wk \sin\theta \sin\varphi,$$
$$I_3\dot{\omega}_3 = 0, \quad \omega_3 = \text{const.} = n.$$

The integral of kinetic energy and work is

$$I_1\omega_1^2 + I_1\omega_2^2 + I_3\omega_3^2 = 2Wk(\cos\theta_0 - \cos\theta) + 2T_0,$$

θ_0 and T_0 being the initial values of the angle $zOz_1 = \theta$ and the kinetic energy T.

The angular momentum about the axis Oz being constant we have

$$I_1\omega_1 \sin\theta \sin\varphi + I_1\omega_2 \sin\theta \cos\varphi + I_3 n \cos\theta = \text{const.} = h_z.$$

If ω_1 and ω_2 be replaced by their values (12), Arts. 435, the two first integrals become

$$I_1(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) = 2Wk(\cos\theta_0 - \cos\theta) - I_3 n^2 + 2T_0,$$
$$I_1\dot{\psi} \sin^2\theta = - I_3 n \cos\theta + h_z;$$

eliminating $\dot{\psi}$ we have for the determination of θ:

$$I_1\dot{\theta}^2 = 2Wk(\cos\theta_0 - \cos\theta) - I_3 n^2 + 2T_0 - \frac{(h_z - I_3 n \cos\theta)^2}{I_1 \sin^2\theta},$$

or introducing $\cos\theta = u$ as new variable:

$$\dot{u}^2 = \left[\frac{2Wk}{I_1}(u_0 - u) + \frac{2T_0 - I_3 n^2}{I_1}\right](1 - u^2) - \frac{1}{I_1^2}(h_z - I_3 n u)^2.$$

Having found u from this equation we have for ψ:

$$\dot{\psi} = \frac{h_z - I_3nu}{I_1(1 - u^2)};$$

and then φ can be found from the third equation (12) which gives

$$\dot{\varphi} = n - \dot{\psi}u = n - \frac{1}{I_1}\frac{h_z - I_3nu}{1 - u^2}u.$$

448. To discuss the equation for u let us put
$I_1[2Wk(u_0 - u) + 2T_0 - I_3n^2](1 - u^2) - (h_z - I_3nu)^2 = f(u)$
so that

$$I_1\dot{u} = \pm\sqrt{f(u)}.$$

As $f(-1) < 0$, $f(u_0) > 0$ (because initially $\dot{u}$ is real), $f(1) < 0$, $f(\infty) > 0$, the cubic $f(u)$ has three real roots, say u_1, u_2, u_3, such that

$$-1 < u_1 < u_0 < u_2 < 1 < u_3 < \infty.$$

For the time we have

$$t = \pm\sqrt{\frac{I_1}{2Wk}}\int\frac{du}{\sqrt{(u - u_1)(u - u_2)(u - u_3)}},$$

the plus or minus sign being used according as du is positive or negative. As $u = \cos\theta$ must lie between -1 and $+1$ it oscillates between its least value u_1 and its greatest value u_2; *i. e.* the axis Oz_1 oscillates between its greatest inclination θ_1 and its least inclination θ_2 to Oz.

449. Suppose, in particular, that the body is initially given a spin about the third principal axis OZ_1 so that $\omega_1 = 0$, $\omega_2 = 0$ for $t = 0$. We may take the axes of reference so that $\varphi = 0$ and $\psi = 0$ for $t = 0$. We then have since h_z is constant:

$$h_z = I_3n\ \cos\theta_0,$$

and

$$f(u) = (u_0 - u)[2I_1Wk(1 - u^2) - I_3^2n^2(u_0 - u)].$$

When $u_0 < u < 1$, $f(u)$ is clearly negative; it is therefore u_2 which is equal to u_0. Hence, at the beginning of the motion u diminishes; in other words, θ_0 is the minimum inclination of the axis OZ_1 to the vertical OZ.

450. The centroid G describes a spherical curve; its projection on the horizontal XY-plane lies between the circles of radii $k\sqrt{1-u_1^2}$ and $k\sqrt{1-u_2^2}$ about O. The co-ordinates x, y of the projection of the centroid on the XY-plane are

$$x = k\sqrt{1-u^2}\sin\psi, \quad y = -k\sqrt{1-u^2}\cos\psi.$$

To determine the direction in which the curve approaches the bounding circles let us determine the angle μ between the radius vector ρ and the tangent to the curve. We have

$$\tan\mu = \frac{\rho}{\frac{d\rho}{d\psi}} = \pm\frac{\sqrt{1-u^2}}{\frac{d}{d\psi}\sqrt{1-u^2}} = \mp\frac{1-u^2}{u}\frac{d\psi}{du}.$$

Now by Arts. 447 and 449

$$\dot\psi = \frac{d\psi}{du}\dot u = \frac{I_3 n}{I_1}\frac{u_0-u}{1-u^2};$$

hence

$$\tan\mu = \mp\frac{I_3 n}{I_1}\frac{u_0-u}{u\dot u}.$$

As $I_1\dot u = \pm\sqrt{f(u)}$ (Art. 448) we find

$$\tan\mu = I_3 n\frac{u_0-u}{u\sqrt{f(u)}} = \frac{I_3 n\sqrt{u_0-u}}{u\sqrt{2WkI_1(u-u_1)(u-u_3)}}.$$

This shows that $\tan\mu$ becomes infinite for $u = u_1$ and zero for $u = u_0 = u_2$. The curve meets therefore the inner circle at right angles (with a cusp) and touches the outer bounding circle. It is in general not a closed curve.

451. The expressions for θ, φ, ψ as functions of t assume a simple form if we suppose the initial angular velocity n about OZ_1 to be very large (Art. 446). In this case the equation (Art. 449)

$$f(u) = (u_0 - u)[2I_1Wk(1 - u^2) - I_3^2n^2(u_0 - u)]=0$$

has its root u_1 nearly equal to u_0 so that the angle θ differs but little from θ_0. Hence if we put $\theta = \theta_0 + \nu$, ν will be small. This gives $\cos\theta = \cos\theta_0 - \nu\sin\theta_0$, *i. e.*

$$\nu = \frac{\cos\theta_0 - \cos\theta}{\sin\theta_0}, \sin\theta = \sin(\theta_0 + \nu) = \sin\theta_0 + \nu\cos\theta_0.$$

Substituting these values in the equation for θ (Art. 447) we find

$$I_1^2\dot{\theta}^2 = 2WkI_1\nu\sin\theta_0 - I_3^2n^2\frac{\nu^2\sin^2\theta_0}{(\sin\theta_0 + \nu\cos\theta_0)^2},$$

or neglecting the term $\nu\cos\theta_0$ in comparison with $\sin\theta_0$:

$$I_1\dot{\theta} = \sqrt{2WkI_1\nu\sin\theta_0 - I_3^2n^2\nu^2}.$$

As $\dot{\theta} = \dot{\nu}$ we find upon integration

$$t = \frac{I_1}{I_3n}\operatorname{versin}^{-1}\frac{\nu}{a}, \quad \text{where } a = \frac{WkI_1\sin\theta_0}{I_3^2n^2},$$

and hence

$$\theta = \theta_0 + \nu = \theta_0 + a\left(1 - \cos\frac{I_3n}{I_1}t\right) = \theta_0 + 2a\sin^2\frac{I_3n}{2I_1}t.$$

The variation ν in the value of θ is called the **nutation**; it is periodic, of period $2\pi I_1/I_3n$.

452. By Arts. 447 and 449,

$$\dot{\psi} = \frac{I_3n}{I_1}\frac{\cos\theta_0 - \cos\theta}{\sin^2\theta} = \frac{I_3n}{I_1}\frac{\nu}{\sin\theta_0},$$

where (Art. 451)

$$\nu = a\left(1 - \cos\frac{I_3 n}{I_1}t\right).$$

Hence, integrating and observing that $\psi = 0$ for $t = 0$:

$$\psi = \frac{I_3 n a}{I_1 \sin\theta_0}t - \frac{a}{\sin\theta_0}\sin\frac{I_3 n}{I_1}t.$$

Thus the first term of ψ increases uniformly with the time, while the second is periodic. The angular velocity $\dot{\psi}$ is the *velocity of precession* (Art. 445).

453. For φ we have by Art. 447:

$$\dot{\varphi} = n - \frac{I_3 n}{I_1}\frac{\cos\theta_0 - \cos\theta}{\sin\theta}\cot\theta = n - \frac{I_3 n}{I_1}\nu\cot\theta_0$$

$$= n - \frac{I_3 n}{I_1}\cot\theta_0 \cdot a\left(1 - \cos\frac{I_3 n}{I_1}t\right);$$

hence

$$\varphi = \left(n - \frac{I_3 n}{I_1}a\cot\theta_0\right)t + a\cot\theta_0\sin\frac{I_3 n}{I_1}t.$$

454. Let us finally inquire into the conditions under which the top while spinning about its axis OZ_1 may keep its inclination $\theta = ZOZ_1$ to the vertical constant. A motion of this kind is often spoken of as *stable*, or steady.

As θ is to be constant we find from Art. 447 that the velocity of precession,

$$\dot{\psi} = \frac{h_z - I_3 n\cos\theta}{I_1 \sin^2\theta}$$

remains constant, say $= \dot{\psi}_0$; and similarly the velocity

$$\dot{\varphi} = n - \dot{\psi}\cos\theta$$

remains constant, say $= \dot{\varphi}_0$. The motion is therefore a regular precession (Art. 445).

The angular velocity ω, at any instant, has the components $\dot{\varphi}_0$ along OZ_1 (Fig. 89) and $\dot{\psi}_0$ along OZ; let us resolve it along OZ_1 and the perpendicular OP_1 to OZ_1 in the plane ZOZ_1; the components will be $\dot{\varphi}_0 + \dot{\psi}_0 \cos\theta$ along OZ_1 and $\dot{\psi}_0 \sin\theta$

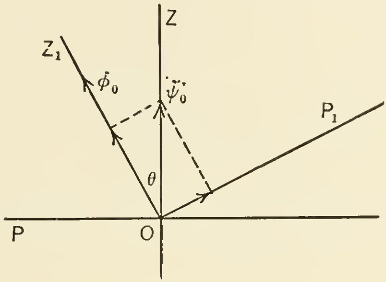

Fig. 89.

along OP_1. As the moment of inertia about OZ_1 is I_3 and that about any perpendicular to OZ_1 is I_1, the angular momentum about OZ_1 is $I_3(\dot{\varphi}_0 + \dot{\psi}_0 \cos\theta)$, while that about OP_1 is $I_1\dot{\psi}_0 \sin\theta$. Hence the angular momenta about OZ and the perpendicular OP to OZ in the plane ZOZ_1 are

$$I_3(\dot{\varphi}_0 + \dot{\psi}_0 \cos\theta) \cos\theta + I_1\dot{\psi}_0 \sin^2\theta,$$

$$I_3(\dot{\varphi}_0 + \dot{\psi}_0 \cos\theta) \sin\theta - I_1\dot{\psi}_0 \sin\theta \cos\theta.$$

The former component is constant; the latter, about OP, receives in the element of time the increment

$$[I_3(\dot{\varphi}_0 + \dot{\psi}_0 \cos\theta) \sin\theta - I_1\dot{\psi}_0 \sin\theta \cos\theta](\dot{\psi}_0 + \dot{\varphi}_0 \cos\theta)dt.$$

If the motion is to be steady this increment must just equal the angular momentum about OP imparted to the body by the force of gravity in the time element, *i. e.* to $Wk \sin\theta \, dt$. Hence the condition

$$[I_3(\dot{\varphi}_0+\dot{\psi}_0 \cos\theta) \sin\theta - I_1\dot{\psi}_0 \sin\theta \cos\theta](\dot{\psi}_0+\dot{\varphi}_0 \cos\theta) = Wk \sin\theta.$$

This requires either $\sin\theta = 0$ which would mean that the axis of the top is vertical, or

$$[I_3(\dot{\varphi}_0 + \dot{\psi}_0 \cos\theta) - I_1\dot{\psi}_0 \cos\theta](\dot{\psi}_0 + \dot{\varphi}_0 \cos\theta) = Wk.$$

For given values of $\dot{\varphi}_0$ and $\dot{\psi}_0$ this condition can in general be satisfied by two different values of $\cos\theta$ since the equation is quadratic in $\cos\theta$.

For a further study of the motion of tops and gyroscopes the following works may be consulted: H. Crabtree, An elementary treatment of the theory of spinning tops and gyroscopic motion, London, Longmans, 1909; A. G. Webster, The dynamics of particles, etc., Leipzig, Teubner, 1904; F. Klein und A. Sommerfeld, Ueber die Theorie des Kreisels, Leipzig, Teubner, 1897–1910.

CHAPTER XIX.

RELATIVE MOTION.

455. We shall here consider only the motion of a particle relatively to a rigid body B having a given motion with respect to fixed axes. By the theorem of Coriolis (Art. 150), the absolute acceleration j of the particle is the resultant of the body acceleration j_b, the complementary acceleration $j_c = 2\omega v_r \cos(\omega, v_r)$, and the relative acceleration j_r:

$$j = j_b + j_c + j_r.$$

If m is the mass of the particle, F the resultant of the given forces acting upon it, its equation of motion is $mj = F$. Hence, multiplying the equation of Coriolis by m and putting

$$-mj_b = F_b, \quad -mj_c = F_c,$$

we find

$$mj_r = F + F_b + F_c.$$

This vector equation gives by projection on the moving axes O_1x_1, O_1y_1, O_1z_1, rigidly connected with the body of reference B:

$$\begin{aligned} m\ddot{x}_1 &= X + X_b + X_c, \\ m\ddot{y}_1 &= Y + Y_b + Y_c, \\ m\ddot{z}_1 &= Z + Z_b + Z_c. \end{aligned} \tag{1}$$

Here X, Y, Z are the components, along the moving axes, of the resultant F of all the given forces acting on the particle. X_b Y_b, Z_b, are the components, along the same axes, of $F_b = -mj_b$, where m is the mass of the particle and j_b the acceleration of that point of the body B with which

the particle happens to coincide at the instant considered; F_b may be called the *body-force*. X_c, Y_c, Z_c are the components of the *complementary force* $F_c = -mj_c$, where j_c is a vector of length $2\omega v_r \sin(\omega, v_r)$, at right angles both to the rotor ω of the body B and to the relative velocity v_r of the particle with respect to B.

Hence we may say that *the equations of the relative motion of the particle m*, *i. e.* of its motion as it would appear to an observer moving with the body of reference B, *are formed like the equations of absolute motion, except that to the given forces acting on the particle must be added the body-force and the complementary force.*

456. It may be noted that the body-force $F_b = -mj_b$ vanishes only when the point of B with which the particle coincides moves uniformly in a straight line, and that $F_c = -2m\omega v_r \sin(\omega, v_r)$ vanishes:

(*a*) when $\omega = 0$, *i. e.* when the body B has a motion of translation;

(*b*) when $v_r = 0$, *i. e.* when the particle is in relative rest;

(*c*) when $\sin(\omega, v_r) = 0$, *i. e.* when the relative velocity v_r of the particle is parallel to the rotor ω, *i. e.* to the instantaneous axis of B.

The principle of kinetic energy and work gives

$$\tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 = \int_{P_0}^{P}[(X + X_b)dx_1 + (Y + Y_b)dy_1 + (Z + Z_b)dz_1]$$

since the work of the complementary force F_c which by definition is normal to the velocity v_r is always zero.

457. *Motion and rest relatively to a body B rotating uniformly about a fixed axis.*

If P (Fig. 90) be that point of B at which the particle m is situated at the time t, $OP = r$ its distance from the fixed axis (through O), the acceleration of P is $j_b = -\omega^2 r$. Hence $F_b = m\omega^2 r$ is directed along OP away from the axis; *i. e.* the body force F_b is in this case what is commonly called the *centrifugal force*.

If, in particular, the particle is absolutely at rest, its relative velocity v_r, *i. e.* the velocity which it appears to have to an observer at P moving with the body B, is equal and opposite to the velocity $v_b = \omega r$ of the point P of B. As regards the accelerations, observe that $j_b = -\omega^2 r$ and,

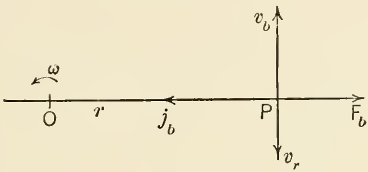

Fig. 90.

since $\measuredangle(\omega, v_r) = \frac{1}{2}\pi$, $j_c = 2\omega v_r = 2\omega^2 r$. The sense of j_c is away from the axis, *i. e.* opposite to j_b. The apparent motion of the particle is a uniform rotation about the fixed axis opposite to that of the body; hence the relative acceleration is $j_r = -\omega^2 r$. The absolute acceleration j is therefore $= j_r + j_b + j_c = -\omega^2 r - \omega^2 r + 2\omega^2 r = 0$, as it should be.

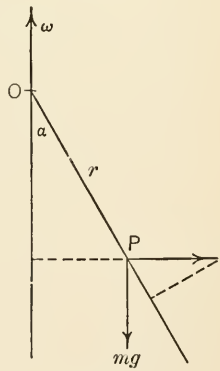

Fig. 91.

458. *Motion of a heavy particle m on a straight line turning uniformly about a vertical axis whose downward direction is met by the line at a constant angle* $\alpha < \frac{1}{2}\pi$ (Fig. 91).

Taking as origin the intersection O of the line with the axis we have $v_r = \dot{r}$ where $r = OP$. The complementary force is taken up by the reaction of the tube.

The components of the weight mg and of the body-force $m\omega^2 r \sin\alpha$ along OP are $mg \cos\alpha$ and $m\omega^2 r \sin^2\alpha$; hence the equation of relative motion:

$$\ddot{r} = g \cos\alpha + \omega^2 r \sin^2\alpha$$

Putting $r + g \cos\alpha/\omega^2 \sin^2\alpha = u$ we have

$$\ddot{u} = (\omega \sin\alpha)^2 u,$$

whence

$$u = r + \frac{g \cos\alpha}{\omega^2 \sin^2\alpha} = C_1 e^{(\omega \sin\alpha)t} + C_2 e^{-(\omega \sin\alpha)t}.$$

If the particle starts from rest at O (or rather from a point very near to O) we find

$$C_1 = C_2 = \frac{1}{2}\frac{g\cos\alpha}{\omega^2\sin^2\alpha};$$

hence

$$r = \frac{g\cos\alpha}{2\omega^2\sin^2\alpha}\left(e^{\frac{1}{2}\omega\sin\alpha\cdot t} - e^{-\frac{1}{2}\omega\sin\alpha\cdot t}\right)^2.$$

For the projection of the path on the horizontal plane we have $\rho = \theta\sin\alpha$, $\theta = \omega t$; hence the projection of the absolute path on the horizontal plane is

$$\rho = \frac{g}{2\omega^2}\cot\alpha\left(e^{\frac{1}{2}\sin\alpha\cdot\theta} - e^{-\frac{1}{2}\sin\alpha\cdot\theta}\right)^2,$$

which represents a spiral.

459. *Motion of a particle relative to the earth, near its surface.*

The earth's motion of translation (which is not uniform) need not be considered since the forces affecting it act on the particle just as

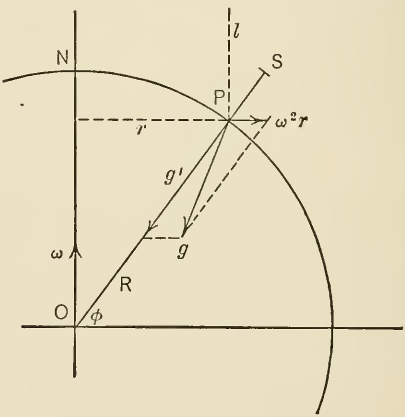

Fig. 92.

they do on the earth and hence do not affect the relative motion. The earth can therefore be regarded as rotating uniformly about a fixed axis; the slight variation of direction of the axis may be neglected.

The angular velocity of the earth is

$$\omega = \frac{2\pi}{86\,164.1} = 0.000\,072\,92 \text{ rad./sec.},$$

the sidereal day having 86 164.1 sec. of mean time.

The *body-force* is simply the centrifugal force (Art. 458) $m\omega^2 r = m\omega^2 R \cos\phi$, where R is the earth's radius and ϕ the latitude.

In most problems of relative motion near the earth's surface the introduction of this centrifugal force is unnecessary. This is best seen by considering a particle at relative rest, say the bob of a pendulum hanging at rest (Fig. 92). Let P be the bob, S the point of suspension, O the earth's center, $OP = R$ the earth's radius, $r = R\cos\phi$ the radius of the parallel in latitude ϕ.

As $v_r = 0$, the complementary force is zero; hence the only forces to be considered are the centrifugal force $m\omega^2 r$, the tension of the rod along PS, and the earth's attraction which is directed along PO if

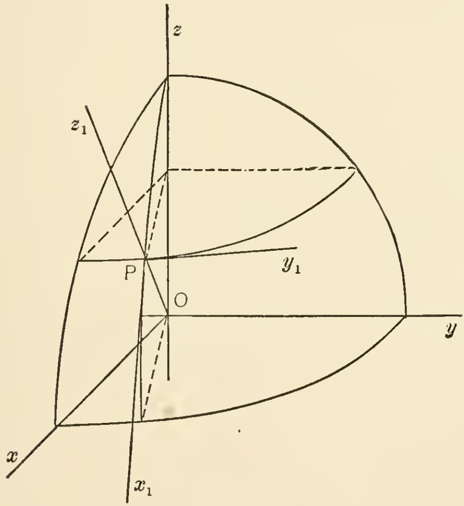

Fig. 93.

we regard the earth as composed of homogeneous spherical layers. Hence the tension of the rod must balance the resultant of the centrifugal force and the attraction. But this resultant is due precisely to the actually observed acceleration g of falling bodies since this includes the combined effect of centrifugal force and attraction.

The *complementary force*, $-2m\omega v_r \sin\alpha$, where α is the angle between the relative velocity v_r and the earth's axis (northward) is at right angles to the plane of the angle α. We take the earth's center O as origin of

the fixed axes and Oz toward the north (Fig. 93); the origin of the moving axes at any point P (in latitude ϕ) on the earth's surface, Pz_1 vertical, Px_1 tangent to the meridian southward, and hence Py_1 tangent to the parallel eastward.

We then have:

$$\omega_1 = \omega\cos(\pi - \phi) = -\omega\cos\phi, \quad \omega_2 = 0, \quad \omega_3 = \omega\sin\phi.$$

Hence the components of the complementary acceleration j_c are

$$\begin{aligned} 2(\omega_2\dot{z}_1 - \omega_3\dot{y}_1) &= -2\omega\dot{y}_1\sin\phi, \\ 2(\omega_3\dot{x}_1 - \omega_1\dot{z}_1) &= 2\omega(\dot{x}_1\sin\phi + \dot{z}_1\cos\phi), \\ 2(\omega_1\dot{y}_1 - \omega_2\dot{x}_1) &= -2\omega\dot{y}_1\cos\phi. \end{aligned}$$

The components of the complementary force F_c along the moving axes are therefore:

$$\begin{aligned} X_c &= 2m\omega\dot{y}_1\sin\phi, \\ Y_c &= -2m\omega(\dot{z}_1\cos\phi + \dot{x}_1\sin\phi), \\ Z_c &= 2m\omega\dot{y}_1\cos\phi. \end{aligned}$$

460. *Relative motion of a heavy particle on a smooth horizontal plane.* The centrifugal force being taken into account by using the observed value of g (Art. 461) the equations of the relative motion are

$$\ddot{x}_1 = 2\omega_3\dot{y}_1, \quad \ddot{y}_1 = 2(\omega_1\dot{z}_1 - \omega_3\dot{x}_1), \quad \ddot{z}_1 = \frac{N}{m} - g - 2\omega_1\dot{y}_1,$$

where N is the normal (*i. e.* vertical) reaction of the plane. As z_1 and $\dot{z}_1$ are constantly zero, the equations reduce to

$$\ddot{x}_1 = 2\omega_3\dot{y}_1, \quad \ddot{y}_1 = -2\omega_3\dot{x}_1, \quad N = m(g + 2\omega_1\dot{y}_1),$$

where $\omega_1 = -\omega\cos\phi$, $\omega_3 = \omega\sin\phi$. The third equation determines N as soon as $\dot{y}_1$ has been found from the first two. The principle of kinetic energy and work gives

$$\tfrac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2) = \text{const.}$$

Hence *the relative or apparent velocity v_r is constant.*

Assuming the particle to start from the origin P we find by integrating each of the two equations by itself:

$$\dot{x}_1 = \dot{x}_0 + 2\omega_3 y_1, \quad \dot{y}_1 = \dot{y}_0 - 2\omega_3 x_1;$$

as $\dot{x}_1^2 + \dot{y}_1^2 = v_r^2 = v_0^2 = \dot{x}_0^2 + \dot{y}_0^2$ we find as *equation of the path*:

$$0 = 4\omega_3\dot{x}_0 y_1 - 4\omega_3\dot{y}_0 x_1 + 4\omega_3^2(y_1^2 + x_1^2),$$

i. e.

$$\left(x_1 - \frac{\dot{y}_0}{2\omega_3}\right)^2 + \left(y + \frac{\dot{x}_0}{2\omega_3}\right)^2 = \left(\frac{v_0}{2\omega_3}\right)^2,$$

a circle tangent to the initial velocity in the horizontal plane. The center C (Fig. 94) lies on the perpendicular to v_0 through P, to the

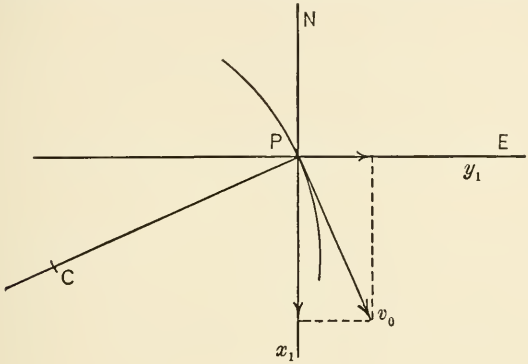

Fig. 94.

right of an observer at P looking in the direction of v_0, in the northern hemisphere, *i. e.* for positive ϕ, to the left in the southern hemisphere.

Thus *the particle deviates to the right in the northern, to the left in the southern hemisphere.*

The radius of the circle is very large since ω is very small. Thus, for $\phi = 30°$ we have for this radius

$$\frac{v_0}{2\omega_3} = \frac{v_0}{\omega} = 13700\, v_0.$$

461. *Particle falling from rest in vacuo.* The equations of motion are the same as in Art. 460 except that $N = 0$:

$$\ddot{x}_1 = 2\omega_3\dot{y}_1, \quad \ddot{y}_1 = 2\omega_1\dot{z}_1 - 2\omega_3\dot{x}_1, \quad \ddot{z}_1 = -g - 2\omega_1\dot{y}_1.$$

If the starting point be taken as origin, the initial conditions are

$$x_0 = 0, \quad y_0 = 0, \quad z_0 = 0, \quad \dot{x}_0 = 0, \quad \dot{y}_0 = 0, \quad \dot{z}_0 = 0;$$

hence the first integrals are

$$\dot{x}_1 = 2\omega_3 y_1, \quad \dot{y}_1 = -2\omega_3 x_1 + 2\omega_1 z_1, \quad \dot{z}_1 = -gt - 2\omega_1 y_1.$$

The method of successive approximations gives the first approximation

$$\dot{x}_1 = 0, \quad \dot{y}_1 = 0, \quad \dot{z}_1 = -gt,$$

whence

$$x_1 = 0, \quad y_1 = 0, \quad z_1 = -\tfrac{1}{2}gt^2.$$

Substituting these values in the expressions for the velocities we find the second approximation

$$\dot{x}_1 = 0, \quad \dot{y}_1 = g\omega\cos\phi\cdot t^2, \quad \dot{z}_1 = -gt,$$

whence

$$x_1 = 0, \quad y_1 = \tfrac{1}{3}g\omega\cos\phi\cdot t^3, \quad z_1 = -\tfrac{1}{2}gt^2.$$

The third approximation gives

$$x_1 = \tfrac{1}{6}g\omega^2\cos\phi\sin\phi\cdot t^4, \quad y_1 = \tfrac{1}{3}g\omega\cos\phi\cdot t^3,$$
$$z_1 = -\tfrac{1}{2}gt^2 + \tfrac{1}{6}g\omega^2\cos^2\phi\cdot t^4.$$

These formulæ show not only an easterly, but also a southerly deviation; the latter is however proportional to ω^2 while the former is proportional to ω. The last value for z shows that the earth's rotation slightly diminishes the vertical distance fallen through in a given time.

462. The eastern deviation of a falling body and the deviation to the right of a projectile (in the northern hemisphere) would furnish an experimental proof of the rotation of the earth if they could be clearly observed. Experiments on falling bodies, with this purpose in view, have been made repeatedly in the last century and even earlier; and the mean results of certain attempts of this kind are often quoted as confirming the theory. But an examination of the individual results shows these so widely discrepant that no reliance can be placed on their mean. In the case of projectiles, such as rifle bullets, the phenomenon is masked completely by the very much larger deviation arising from the rotation of the projectile and the resistance of the air.

For this reason Foucault's pendulum experiment, first made in 1851, and since often repeated with good success, is of particular interest. On a fixed earth, a pendulum set swinging in a vertical plane would continue to swing in the same plane; on the rotating earth, the plane in which the pendulum swings, remaining fixed in space, must apparently, *i. e.* relatively to the earth, turn about the vertical through the point of suspension, in the sense opposite to that of the earth's rotation, with the angular velocity $\omega\sin\phi$, where ω is the angular velocity of the earth and ϕ the latitude of the place of observation.

463. Foucault's pendulum. It will be convenient to take the point of suspension O as origin, the axis Oz_1 vertically downward, Ox_1 tangent to the meridian northward, and hence Oy_1 tangent to the parallel eastward. The forces acting on the bob are its weight mg, the tension N of the suspending wire, and the complementary force F_c whose components are, since $\omega_1 = \omega\cos\phi$, $\omega_3 = -\omega\sin\phi$:

$$X_c = -2m\omega\dot{y}_1\sin\phi,\quad Y_c = 2m\omega(\dot{x}_1\sin\phi + \dot{z}_1\cos\phi),\quad Z_c = -2m\omega\dot{y}_1\cos\phi.$$

If $l = \sqrt{x_1^2 + y_1^2 + z_1^2}$ is the length of the wire the equations of motion are:

$$\ddot{x}_1 = -\frac{N}{m}\frac{x_1}{l} - 2\omega\dot{y}_1\sin\phi,$$

$$\ddot{y}_1 = -\frac{N}{m}\frac{y_1}{l} + 2\omega\dot{x}_1\sin\phi + 2\omega\dot{z}_1\cos\phi,$$

$$\ddot{z}_1 = -\frac{N}{m}\frac{z_1}{l} - 2\omega\dot{y}_1\cos\phi + g,$$

with the condition $l^2 = x_1^2 + y_1^2 + z_1^2$.

The general integration of these equations would present serious difficulties. But for small oscillations we have

$$\frac{z_1}{l} = \left(1 - \frac{x_1^2 + y_1^2}{l^2}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\frac{x_1^2 + y_1^2}{l^2} - \frac{1}{8}\left(\frac{x_1^2 + y_1^2}{l^2}\right)^2 - \cdots.$$

As x_1, y_1, $\dot{x}_1$, $\dot{y}_1$, $\ddot{x}_1$, $\ddot{y}_1$, are small, say of the first order, $\dot{z}_1$ and $\ddot{z}_1$ will be small of the second order; for we have $z_1^2 = l^2 - x_1^2 - y_1^2$, $z_1\dot{z}_1 = -x_1\dot{x}_1 - y_1\dot{y}_1$, $z_1\ddot{z}_1 + \dot{z}_1^2 = -x_1\ddot{x}_1 - y_1\ddot{y}_1 - \dot{x}_1^2 - \dot{y}_1^2$, whence

$$\ddot{z}_1 = -\frac{1}{z_1}(x_1\ddot{x}_1 + y_1\ddot{y}_1 + \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2)$$

$$= -\frac{1}{z_1}(x_1\ddot{x}_1 + y_1\ddot{y}_1 + \dot{x}_1^2 + \dot{y}_1^2) - \frac{1}{z_1^3}(x_1\dot{x}_1 + y_1\dot{y}_1)^2.$$

We take therefore as first approximation $\ddot{z}_1 = 0$, $z_1 = l$ so that the third equation of motion reduces to

$$N = m(g - 2\omega\dot{y}_1\cos\phi).$$

Substituting this value in the first two equations and neglecting terms of the second order we find if we write ω' for $\omega\sin\phi$:

$$\ddot{x}_1 + 2\omega'\dot{y}_1 + \frac{g}{l}x_1 = 0,\quad \ddot{y}_1 - 2\omega'\dot{x}_1 + \frac{g}{l}y_1 = 0.$$

Multiplying by y_1, x_1 and subtracting we have

$$x_1\ddot{y}_1 - y_1\ddot{x}_1 = 2\omega'(x_1\dot{x}_1 + y_1\dot{y}_1),$$

that is:

$$\frac{d}{dt}(x_1\dot{y}_1 - y_1\dot{x}_1) = \omega' \frac{d}{dt}(x_1^2 + y_1^2).$$

Hence, integrating and putting $x_1 = r\cos\theta$, $y_1 = r\sin\theta$:

$$r^2\dot{\theta} = \omega' r^2 + C.$$

If $r = 0$, $\dot{\theta} = 0$ for $t = 0$ we have $C = 0$ so that

$$\dot{\theta} = \omega',$$

and hence

$$\theta = \theta_0 + \omega' t.$$

This means that the apparent motion consists of the rotation of the plane in which the pendulum swings about the vertical with the constant angular velocity $\omega' = \omega\sin\phi$. The plane makes one complete revolution in the time $T = 2\pi/\omega\sin\phi$.

464. In theoretical mechanics the motion of any particle, rigid body, or variable system is referred ultimately to a reference system (co-ordinate trihedral) which is regarded as fixed. In applying mechanics to the study of physical phenomena we meet with the difficulty that in nature no absolutely fixed object is to be found. For motions in the vicinity of any particular point on the earth's surface we regard the earth as fixed. In astronomy, the motions of the planets are referred to the sun as if it were a fixed center; and the motion of the solar system is referred to the fixed stars. But it is well known that even the so-called fixed stars have their proper motions. Thus in all these cases we are merely dealing with *relative* motions.

465. It should be observed that the differential equations of motion of a particle are the same whether the reference system is at rest or has a *rectilinear uniform translation.* In other words, these differential equations *admit* such a translation. For, if for x, y, z we substitute $x_1 + v_1t$, $y_1 + v_2t$, $z_1 + v_3t$, where v_1, v_2, v_3 are the constant components of the velocity of translation, we have $\ddot{x} = \ddot{x}_1$, $\ddot{y} = \ddot{y}_1$, $\ddot{z} = \ddot{z}_1$.

466. Other difficulties in the fundamental concepts of mechanics concern the idea of *time.*

All our measurements of time are based ultimately on the assumption that the earth's rotation is strictly uniform. That this assumption,

which can not be verified directly, must be true to a very high degree of approximation may be inferred from the agreement of astronomical predictions with actual occurrences.

Another question, and one that has been much discussed in recent years, arises from the difficulty of defining the simultaneity of two events occurring at places in motion relatively to the observer or observed by persons in motion relative to each other. Consider an observer at P at different distances from the points A, B. If the times it takes light to travel the distances AP and BP are t_1 and t_2, then flashes of light occurring simultaneously at A and B will appear to the observer to happen at different times, the difference being $|t_1 - t_2|$.

Again, if the observer is in motion, *e. g.* moving toward A with velocity v, a flash given at A when the observer is at P will appear to him to happen at a time $AP/(V + v)$ after it actually occurred (V being the velocity of light). Thus the statement that two events are simultaneous does not have a definite meaning unless the position and motion of the observer are known.

In mechanics we deal ordinarily with velocities which are very small in comparison with the velocity of light. By regarding the velocity of light as infinite, the difficulty would disappear. In the electron theory where the moving electron has a velocity comparable with that of light the idea becomes of importance.

CHAPTER XX.

MOTION OF A SYSTEM OF PARTICLES.

1. Free system.

467. A system consisting of any finite number of particles is called *free* if the co-ordinates of the particles are subject to no conditions, whether these be expressed by equations or inequalities. The forces acting on any one of the particles are distinguished as *internal* or *external* according as they are exerted by the other particles of the system or proceed from sources outside of the system.

Examples of such systems we have on the one hand in celestial mechanics, the most simple case being the problem of two bodies (Arts. 321–327), on the other in the kinetic theory of gases where the particles are the molecules of the gas.

468. Let X, Y, Z be the rectangular components of the resultant of all the external and internal forces acting on any one of the n particles; m the mass and x, y, z the co-ordinates of the particle; then the equations of motion of this particle are

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z. \tag{1}$$

There are 3 such equations for each particle and hence $3n$ for the whole system; they express the equilibrium of the external, internal, and reversed effective forces.

If we assume that the internal forces occur only in pairs of equal and opposite forces between the particles, depending only on the mutual positions and not on the velocities of the particles, almost all the results developed in Chapter XV for

the system of particles constituting a rigid body will hold for the free system, except that we have now $3n$, instead of merely six, equations.

469. D'Alembert's principle is expressed by the equation

$$\Sigma(-m\ddot{x}+X)\,\delta x+\Sigma(-m\ddot{y}+Y)\,\delta y+\Sigma(-m\ddot{z}+Z)\,\delta z=0, \quad (2)$$

in which δx, δy, δz are the components of an arbitrary displacement δs of the particle m. As the $3n$ virtual displacements are independent of each other this equation (2) is equivalent to the $3n$ equations (1).

If this equation be written in the form

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma(X\delta x + Y\delta y + Z\delta z) \quad (2')$$

the right-hand member will contain only the external forces owing to the assumption (Art. 468) concerning the internal forces.

As there are no constraints or conditions we may select for δs the actual displacement ds of every particle; the equation

$$\Sigma m(\ddot{x}dx + \ddot{y}dy + \ddot{z}dz) = \Sigma(Xdx + Ydy + Zdz)$$

then gives upon integration the *equation of kinetic energy and work*:

$$\Sigma\tfrac{1}{2}mv^2 - \Sigma\tfrac{1}{2}mv_0^2 = \Sigma\int_0^t(Xdx + Ydy + Zdz). \quad (3)$$

If in particular, there exists a force function or potential U for the forces X, Y, Z, the system is said to be *conservative.* We then have

$$\Sigma(Xdx + Ydy + Zdz) = dU,$$

so that (3) becomes in the usual notation ($V = -U$):

$$T + V = T_0 + V_0 = \text{const.};$$

this expresses the **principle of the conservation of energy.**

470. A system of n particles possesses a centroid whose co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ at any instant are given by the equations

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my, \quad M\bar{z} = \Sigma mz,$$

where $M = \Sigma m$. The *principles of the conservation of linear and angular momentum* (Arts. 363, 366) are found to hold just as for a rigid body.

Thus, in the case of the solar system, if the action of the fixed stars be neglected, the centroid of the system must move uniformly in a straight line and there exists an "invariable plane" (Art. 367).

2. Constrained system.

471. In the case of a system of particles subject to constraints or conditions, we may try to replace the conditions by *constraining forces* or *reactions* after the introduction of which the system can be treated as free. The equations of motion of the particle m will then again have the form (1), Art. 468; but the right-hand members now contain the unknown reactions. The principle of virtual work gives d'Alembert's equation (2), Art. 469; and the virtual displacement can often be selected so that the unknown constraining forces will do no work and hence will not appear in equation (2). This constitutes the main advantage of d'Alembert's principle.

472. Before proceeding it may be well to indicate here the considerations by which d'Alembert himself (and, in more exact language, Poisson) explained his celebrated principle.

Any particle m of the system is acted upon at any time t by two kinds of forces, the *given* external and internal forces, whose resultant we denote by F (Fig. 95), and the internal reactions and constraining forces whose resultant we call F'. The resultant of F and F' must be geometrically equal to the effective force mj, where j is the acceleration of the particle at the time t.

Now, if we introduce at m the equal and opposite forces mj, $-mj$, the motion of the particle is not affected. But we can now replace F and $-mj$ by their resultant F''; and as F, F', $-mj$ are in equilibrium, so are the forces F' and F''; *i. e.* F'' is equal and opposite to F'.

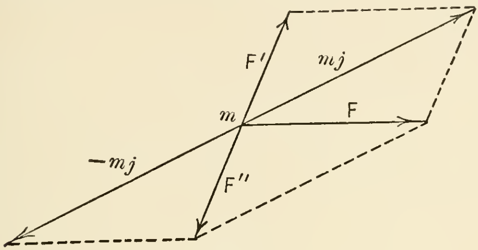

Fig. 95.

The figure shows that F can be resolved into the components mj and F''; the former produces the actual change of motion of the particle while the latter is consumed in overcoming the internal reactions and constraints represented by F'. This component F'' of F is therefore called by d'Alembert the *lost force*. As $F' + F'' = 0$ at every particle of the system, d'Alembert's principle can be expressed by saying that, at every moment during the motion, *the lost forces are in equilibrium with the constraints of the system.*

If the constraints, instead of being expressed by means of forces, are given by equations of condition we may express the same idea by saying that, *owing to the given conditions, the lost forces form a system in equilibrium.*

473. We shall now assume that the constraints or conditions to which the system is subject are expressed by means of equations (the case of conditions expressed by inequalities is excluded) between the co-ordinates x, y, z of the particles and the time t; such systems are called *holonomic.* If the equations contained the velocities, the system would be called non-holonomic.

A simple illustration of the difference between the two is furnished by a sphere moving on a plane. The position of

the sphere can be determined by the co-ordinates x, y of its center and Euler's angles θ, φ, ψ (Art. 434). If the plane is smooth the system is holonomic; if it is so rough as to prevent slipping, $\dot{x}$, $\dot{y}$, $\dot{\theta}$, $\dot{\varphi}$, $\dot{\psi}$ are no longer independent, and the system is therefore non-holonomic.

474. Let there be k conditions

$$\varphi(t, x_1, y_1, z_1, x_2, \cdots) = 0, \quad \psi(t, x_1, y_1, z_1, x_2, \cdots) = 0, \quad \cdots \quad (4)$$

for a holonomic system of n particles. The number of independent equations of motion will be $3n - k$.

For, these equations must express the equilibrium of the given forces, together with the reversed effective forces, under the given conditions; and for this equilibrium it is sufficient that *the virtual work should vanish for any displacement compatible with the conditions*, the work of the reactions and constraining forces being zero for such virtual displacement. In other words, in d'Alembert's equation (2), Art. 469, the constraining forces due to the conditions will not appear if the displacements δx, δy, δz be so selected as to be compatible with the k conditions (4). Now this will be the case if these displacements are made to satisfy the equations that result from differentiating the conditions (4), viz.

$$\Sigma(\varphi_x \delta x + \varphi_y \delta y + \varphi_z \delta z) = 0, \quad \Sigma(\psi_x \delta x + \psi_y \delta y + \psi_z \delta z) = 0, \quad \cdots \quad (5)$$

As in Art. 347, t is regarded as constant in this differentiation. Indeed, when the conditions contain the time, a virtual displacement is *defined* as one satisfying the conditions (5).

475. By means of the k equations (5), k of the $3n$ displacements δx, δy, δz can be eliminated from d'Alembert's equation (2). The remaining $3n - k = m$ displacements are arbitrary; their coefficients must therefore vanish separately; equating them to zero we have the $3n - k = m$

equations of motion of a system of n particles with k conditions.

To do this more systematically we may, as in Arts. 348, 351, use Lagrange's method of indeterminate multipliers: adding the equations (5), multiplied by λ, μ, $\cdots$, to d'Alembert's equation (2), we obtain a single equation in which the k multipliers λ, μ, $\cdots$ can be selected so as to make the coefficients of k of the $3n$ displacements δx, δy, δz vanish. The remaining $3n - k$ displacements being arbitrary their coefficients must likewise vanish. Hence the coefficients of *all* the displacements must be equated to zero, and this gives n sets of 3 equations of the type

$$\begin{aligned} m\ddot{x} &= X + \lambda\varphi_x + \mu\psi_x + \cdots, \\ m\ddot{y} &= Y + \lambda\varphi_y + \mu\psi_y + \cdots, \\ m\ddot{z} &= Z + \lambda\varphi_z + \mu\psi_z + \cdots. \end{aligned} \tag{6}$$

These, together with the equations (4), are sufficient to determine the $3n$ co-ordinates x, y, z and the k multipliers λ, μ, $\cdots$.

It is apparent from the equations (6) that the constraining force acting on the particle m has the components:

$$\begin{aligned} X' &= \lambda\varphi_x + \mu\psi_x + \cdots, \\ Y' &= \lambda\varphi_y + \mu\psi_y + \cdots, \\ Z' &= \lambda\varphi_z + \mu\psi_z + \cdots. \end{aligned}$$

476. If the conditions (4) do not contain the time the actual displacements dx, dy, dz of the particles can be taken as virtual displacements; and d'Alembert's equation then gives the *equation of kinetic energy and work*

$$d\Sigma\tfrac{1}{2}mv^2 = \Sigma(Xdx + Ydy + Zdz). \tag{7}$$

This also follows from the equations (6) after multiplying them by $\dot{x}dt$, $\dot{y}dt$, $\dot{z}dt$ and adding. For, the coefficients of λ, μ, $\cdots$ in the resulting equation, viz. $\Sigma(\varphi_x\dot{x} + \varphi_y\dot{y} + \varphi_z\dot{z})dt$, $\Sigma(\psi_x\dot{x} + \psi_y\dot{y} + \psi_z\dot{z})dt$, $\cdots$ are zero as appears by differ-

entiating the conditions (4) with respect to t. This means that the constraining forces in this case do no work in the actual displacement of the system, as they are all perpendicular to the paths of the particles.

If, however, the conditions (4) contain the time explicitly, their differentiation gives

$$\Sigma(\varphi_x\dot{x}+\varphi_y\dot{y}+\varphi_z\dot{z})+\varphi_t=0, \quad \Sigma(\psi_x\dot{x}+\psi_y\dot{y}+\psi_z\dot{z})+\psi_t=0, \cdots,$$

so that instead of (7) we find:

$$d\Sigma\tfrac{1}{2}mv^2 = \Sigma(Xdx+Ydy+Zdz)-\lambda\varphi_t dt-\mu\psi_t dt-\cdots. \quad (7')$$

477. If the conditions (4) do not contain the time and if, moreover, a force-function $U = -V$ exists for all the forces we find as in Art. 469 the *principle of the conservation of energy:*

$$T+V=T_0+V_0.$$

But, even if no force-function exists, the elementary work $\Sigma(Xdx+Ydy+Zdz)$ is a quantity independent of the co-ordinate system, and $\int_0^t\Sigma(Xdx+Ydy+Zdz)=W$, say, represents the work done by the external and internal forces in the time t; we have therefore:

$$\Sigma\tfrac{1}{2}mv^2-\Sigma\tfrac{1}{2}mv_0^2=W.$$

3. Generalized co-ordinates; Lagrange's equations of motion; Hamilton's principle.

478. As shown in Art. 358, the number of conditions that make a system of n particles *invariable*, *i. e.* make it a free rigid body, is $k = 3n-6$. A free rigid body has therefore $3n-k=6$ independent equations of motion.

A rigid body with a fixed axis (Art. 415) has but 1 degree of freedom and 5 constraints; its motion is given by a single equation.

A rigid body that can turn about and also slide along a fixed axis (Art. 235) has 4 constraints and 2 degrees of freedom; it has 2 equations of motion, its position being determined by 2 co-ordinates, say the angle θ and the distance x measured along the axis.

A rigid body with one fixed point (Art. 233) is an example of an invariable system with 3 constraints and 3 degrees of freedom. Three variables (such as Euler's angles θ, φ, ψ, Art. 434) are necessary and sufficient to determine a particular position, and the number of independent equations of motion is 3.

479. These considerations can be generalized so as to apply to a general variable system of n particles with k holonomic conditions. Such a system is said to have $3n - k = m$ *co-ordinates* because it has $3n - k = m$ independent equations of motion (Art. 474). In other words, in the place of the $3n$ cartesian co-ordinates x, y, z of the n particles, subject to k conditional equations (4), we may introduce $3n - k = m$ *independent* variables, say $q_1, \cdots q_m$, which are selected so as to satisfy the k conditions (4) identically. These vairables are called the **generalized,** or **lagrangian, co-ordinates** of the system (comp. Art. 352).

Suppose, for instance, that the system is subject to only one condition, viz. that the point P_1 of the system should remain on the surface of the ellipsoid

$$\varphi \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 = 0.$$

If we select two new variables q_1, q_2, connected with x_1, y_1, z_1 by the equations

$$x_1 = a \cos q_1, \quad y_1 = b \sin q_1 \cos q_2, \quad z_1 = c \sin q_1 \sin q_2,$$

the condition $\varphi = 0$ is satisfied identically in the new co-ordinates q_1, q_2. Hence, by introducing q_1, q_2 in the place of x_1, y_1, z_1 the condition $\varphi = 0$ is eliminated from the problem.

The motion of a system with m degrees of freedom in ordinary three-dimensional space might be interpreted as the motion of a free particle in a space of m dimensions.

480. The introduction of the generalized co-ordinates $q_1, \cdots q_m$ of a system with $m = 3n - k$ degrees of freedom into the equations of motion (6), Art. 475, is performed just as the corresponding problem in Arts. 353–355.

The cartesian co-ordinates x, y, z of any one of the n particles are given functions of $q_1, \cdots q_m$ and of t so that

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial x}{\partial q_m}\dot{q}_m + \frac{\partial x}{\partial t},$$

with similar expressions for $\dot{y}, \dot{z}$. Hence, on the one hand we have if q denote any one of the co-ordinates $q_1, \cdots q_m$:

$$\frac{\partial \dot{x}}{\partial \dot{q}} = \frac{\partial x}{\partial q}, \quad \frac{\partial \dot{y}}{\partial \dot{q}} = \frac{\partial y}{\partial q}, \quad \frac{\partial \dot{z}}{\partial \dot{q}} = \frac{\partial z}{\partial q}. \tag{8}$$

on the other:

$$\begin{aligned}\frac{\partial \dot{x}}{\partial q} &= \frac{\partial^2 x}{\partial q \partial q_1}\dot{q}_1 + \cdots + \frac{\partial^2 x}{\partial q \partial q_m}\dot{q}_m + \frac{\partial^2 x}{\partial q \partial t} \\ &= \dot{q}_1 \frac{\partial}{\partial q_1}\frac{\partial x}{\partial q} + \cdots + \dot{q}_m \frac{\partial}{\partial q_m}\frac{\partial x}{\partial q} + \frac{\partial}{\partial t}\frac{\partial x}{\partial q} = \frac{d}{dt}\frac{\partial x}{\partial q},\end{aligned}$$

that is:

$$\frac{\partial \dot{x}}{\partial q} = \frac{d}{dt}\frac{\partial x}{\partial q}, \quad \frac{\partial \dot{y}}{\partial q} = \frac{d}{dt}\frac{\partial y}{\partial q}, \quad \frac{\partial \dot{z}}{\partial q} = \frac{d}{dt}\frac{\partial z}{\partial q}. \tag{9}$$

With the aid of the relations (8) and (9) we find for the derivatives of the kinetic energy $T = \Sigma \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$:

$$\frac{\partial T}{\partial q} = \Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial q} + \dot{y}\frac{\partial \dot{y}}{\partial q} + \dot{z}\frac{\partial \dot{z}}{\partial q}\right)$$
$$= \Sigma m\left(\dot{x}\frac{d}{dt}\frac{\partial x}{\partial q} + \dot{y}\frac{d}{dt}\frac{\partial y}{\partial q} + \dot{z}\frac{d}{dt}\frac{\partial z}{\partial q}\right),$$

$$\frac{\partial T}{\partial \dot{q}} = \Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}}\right) = \Sigma m\left(\dot{x}\frac{\partial x}{\partial q} + \dot{y}\frac{\partial y}{\partial q} + \dot{z}\frac{\partial z}{\partial q}\right);$$

hence

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \Sigma m\left(\ddot{x}\frac{\partial x}{\partial q} + \ddot{y}\frac{\partial y}{\partial q} + \ddot{z}\frac{\partial z}{\partial q}\right) + \frac{\partial T}{\partial q}. \qquad (10)$$

Now multiplying the equations of motion (6) by $\partial x/\partial q$, $\partial y/\partial q$, $\partial z/\partial q$ and adding them throughout the whole system we find

$$\Sigma m\left(\ddot{x}\frac{\partial x}{\partial q} + \ddot{y}\frac{\partial y}{\partial q} + \ddot{z}\frac{\partial z}{\partial q}\right) = \Sigma\left(X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q}\right), \qquad (11)$$

the coefficients of λ, μ, $\cdots$ being all zero since, by hypothesis, the new co-ordinates satisfy the conditions (4) identically.

The right-hand member of (11) will be denoted by Q (comp. Art. 353); substituting for the left-hand member its value from (10) we find **Lagrange's equations of motion:**

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q. \qquad (12)$$

As there is one such equation for each of the lagrangian co-ordinates $q_1, \cdots q_m$, their number is $m = 3n - k$. They are obtained from the type (12) by attaching successively the subscripts $1, \cdots m$ to q, $\dot{q}$, and Q.

481. In the particular case of a *conservative system, i. e.* when there exists a force-function U such that

$$\Sigma X = \frac{\partial U}{\partial x}, \quad \Sigma Y = \frac{\partial U}{\partial y}, \quad \Sigma Z = \frac{\partial U}{\partial z},$$

we have $Q = \partial U/\partial q$; the equations of motion then have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \frac{\partial}{\partial q}(T + U). \tag{12'}$$

This relation can be derived directly from the equation (10) by observing that in any infinitesimal displacement the work of the effective forces is equal to the increase of U (or decrease of the potential energy $V = -U$). Now if we vary the co-ordinate q alone by δq, the variations of x, y, z are $(\partial x/\partial q)\delta q$, $(\partial y/\partial q)\delta q$, $(\partial z/\partial q)\delta q$; hence the work of the effective forces $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$ is

$$\delta q \Sigma m\left(\ddot{x}\frac{\partial x}{\partial q} + \ddot{y}\frac{\partial y}{\partial q} + \ddot{z}\frac{\partial z}{\partial q}\right) = \frac{\partial U}{\partial q}\delta q;$$

the first term in the right-hand member of (10) is therefore $= \partial U/\partial q$, and this at once gives (12').

482. Finally, if we denote the function $T + U$, that is, the difference $T - V$ of kinetic and potential energy, by L:

$$L = T + U = T - V,$$

and observe that U is independent of the velocities so that

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}},$$

the equations of motion can be written in the simple form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \tag{12''}$$

in which they depend on a single function. This function L is called the **kinetic potential** (according to Helmholtz) or the *lagrangian function.*

483. To illustrate the use of Lagrange's equations let us derive the equations of motion in polar co-ordinates.

In the case of plane motion we have

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

whence

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{r}} = m\ddot{r}, \quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2.$$

The left-hand member of (12) is therefore $m(\ddot{r} - r\dot{\theta}^2)$. The right-hand member Q is determined by observing that $Q\delta q$ is the work of the forces in the displacement δq. Hence in our case, if R is the resultant force, R_r and R_θ its components along and at right angles to the radius vector r (Art. 269), $Q\delta r = R_r\delta r$, *i. e.* $Q = R_r$. Hence the first equation of motion is

$$m(\ddot{r} - r\dot{\theta}^2) = R_r.$$

To find the second equation we have

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial T}{\partial \theta} = 0;$$

as $Q\delta\theta$ is the work done on the particle when θ varies by $\delta\theta$, *i. e.* in the displacement $r\delta\theta$ at right angles to the radius vector, we have $Q\delta\theta = R_\theta r\delta\theta$; hence the second equation

$$m\frac{d}{dt}(r^2\dot{\theta}) = rR_\theta.$$

484. For polar co-ordinates r, θ, ϕ in three dimensions (Art. 269) we have

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2),$$

and we find:

$$m[\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)] = R_r,$$

$$m\left[\frac{d}{dt}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\,\dot{\varphi}^2\right] = rR_\theta,$$

$$m\frac{d}{dt}(r^2\sin^2\theta\,\dot{\varphi}) = r\sin\theta\,R_\phi.$$

If there exists a force-function U the right-hand members are $\partial U/\partial r$, $\partial U/\partial\theta$, $\partial U/\partial\varphi$.

485. As another example consider the motion, in the vertical xy-plane of two particles P, P' (Fig. 96) of masses m, m', suspended by

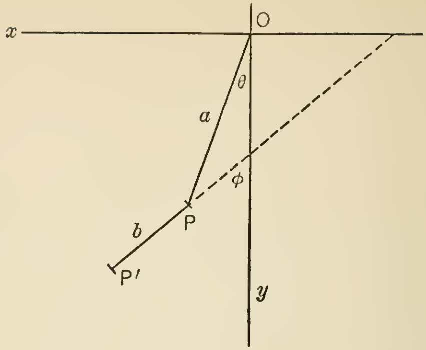

Fig. 96.

weightless rods OP, PP' of lengths a, b. If the co-ordinates of P, P' are x, y and x', y' and the inclinations of OP, PP' to the vertical θ, φ, we have

$$x = a\sin\theta, \quad y = a\cos\theta,$$

$$x' = a\sin\theta + b\sin\varphi, \quad y' = a\cos\theta + b\cos\varphi,$$

whence

$$U = mgy + m'gy' = g[(m + m')a\cos\theta + m'b\cos\varphi].$$

Denoting by v, v' the velocities of P, P' we have

$$v^2 = a^2\dot\theta^2, \quad v'^2 = a^2\dot\theta^2 + b^2\dot\varphi^2 + 2ab\cos(\varphi - \theta)\dot\theta\dot\varphi,$$

whence

$$T = \tfrac{1}{2}[(m + m')a^2\dot\theta^2 + m'b^2\dot\varphi^2 + 2m'ab\cos(\varphi - \theta)\dot\theta\dot\varphi].$$

Lagrange's equations are then found to be

$$\frac{d}{dt}[(m + m')a^2\dot\theta + m'ab\cos(\varphi - \theta)\,\dot\varphi]$$

$$- m'ab\sin(\varphi - \theta)\dot\theta\dot\varphi + g(m + m')a\sin\theta = 0,$$

$$\frac{d}{dt}m'[b^2\dot\varphi + ab\cos(\varphi - \theta)\dot\theta] + m'ab\sin(\varphi - \theta)\dot\theta\dot\varphi + gm'b\sin\varphi = 0.$$

If, in particular, θ, φ, $\dot\theta$, $\dot\varphi$ are so small that their third powers can be neglected the equations reduce to

$$(m + m')a\ddot\theta + m'b\ddot\varphi + (m + m')g\theta = 0,$$

$$a\ddot\theta + b\ddot\varphi + g\varphi = 0.$$

To integrate put $\theta = A\cos rt$, $\varphi = \lambda A\cos rt$, whence

$$(m + m')(g - ar^2) = m'b\lambda r^2, \quad \lambda(g - br^2) = ar^2,$$

$$mabr^4 - (m + m')g(a + b)r^2 + (m + m')g^2 = 0.$$

The last equation regarded as a quadratic in r^2 has real (the discriminant being positive) and positive roots, say r^2 and r_1^2. Corresponding to these roots we have

$$\theta = A \cos rt, \quad \varphi = \frac{Aar^2}{g - br^2} \cos rt,$$

$$\theta = A_1 \cos r_1 t, \quad \varphi = \frac{A_1 a r_1^2}{g - br_1^2} \cos r_1 t.$$

In the same way, by substituting

$$\theta = A \sin rt, \quad \varphi = \lambda A \sin rt$$

we obtain the special solutions

$$\theta = A' \sin rt, \quad \varphi = \frac{A' ar^2}{g - br^2} \sin rt,$$

$$\theta = A_1' \sin r_1 t, \quad \varphi = \frac{A_1' a r_1^2}{g - br_1^2} \sin r_1 t.$$

The general solution is of course the sum of the four special solutions, and the four constants are determined from the initial positions and velocities of m and m'.

486. From Lagrange's equations (12), Art. 480, it is easy to derive **Hamilton's principle.**

Let each of the m equations (12) be multiplied by the infinitesimal displacement, or variation, δq; let the equations be added, multiplied by dt, and integrated from t_1 to t_2:

$$\int_{t_1}^{t_2} \Sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} - Q \right) \delta q dt = 0. \qquad (13)$$

The first term can be transformed by integration by parts; as $d(\delta q)/dt = \delta(dq)/dt$ we have

$$\int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) \delta q dt = \frac{\partial T}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}} \delta \dot{q} dt.$$

If now the variations δq be selected so as to vanish both at the time t_1 and at the time t_2, the first term on the right vanishes at both limits. Hence the equation (13) assumes the form

$$\int_{t_1}^{t_2} \Sigma \left(\frac{\partial T}{\partial \dot{q}}\,\delta\dot{q} + \frac{\partial T}{\partial q}\,\delta q + Q\delta q\right)dt = 0.$$

As $\Sigma\left(\frac{\partial T}{\partial \dot{q}}\,\delta\dot{q} + \frac{\partial T}{\partial q}\,\delta q\right) = \delta T$ and $\Sigma Q\delta q = \delta U$ for a conservative system and $= \delta W$ (the elementary work) for any system, the equation reduces to the simple form

$$\int_{t_1}^{t_2} (\delta T + \delta W)dt = 0 \tag{14}$$

in the general case, and

$$\delta \int_{t_1}^{t_2} (T+U)dt = 0, \text{ or } \delta \int_{t_1}^{t_2} (T-V)dt = 0, \text{ or } \delta \int_{t_1}^{t_2} Ldt = 0, \tag{14$'$}$$

for a conservative system.

487. Hamilton's principle consists in the proposition that the equation (14) or (14′) holds for any virtual displacements of the system that are zero at the times t_1 and t_2.

In the case of a conservative system, where $\delta \int_{t_1}^{t_2} Ldt = 0$, the principle is often expressed briefly by saying that *for the actual motion the mean value of the kinetic potential* $L = T - V$ *in any time* $t_2 - t_1$ *is a minimum as compared with other motions between the same two configurations.* More exactly we can only say that the variation of this mean value, *i. e.* of the integral $\int_{t_1}^{t_2} Ldt$, is zero.

A more complete discussion of Hamilton's principle and of the somewhat similar principle of least (or stationary) action will be found in E. T. Whittaker, Analytical dynamics, Cambridge, University Press, 1904.

ANSWERS.

Art. 7.

(1) (*a*) 5.87; (*b*) 40.62; (*c*) 58.67; (*d*) 25.38; (*e*) 1086.9; (*f*) 82,020; (*g*) 9.84×10^8.

(2) $t = 2(a + b)/(v_1 + v_2)$.

(3) 184,200 M./sec.

(4) 35 M./h.

(5) $\frac{5}{4}$.

(6) $18\frac{3}{4}$.

(7) 10.4.

Art. 9.

(1) (*a*) a; (*b*) $2at+b$; (*c*) $\frac{1}{2}a/\sqrt{t}$; (*d*) $-ak \sin kt$; (*e*) $-ae^{-t}$; (*f*) $\frac{1}{2}a(e^t - e^{-t})$; (*g*) $at(t + 1)$; (*h*) $4at(t^2 - 1)$; (*i*) $at(3t - 2)$; (*j*) $5at(t^3 - 8)$; (*k*) $a(1 + t^2)/(1 - t^2)^2$.

(2) (*a*) $s_0 + v_0t + \frac{1}{2}gt^2$; (*b*) $s_0 - 4at + \frac{1}{3}at^3$; (*c*) $s_0 + a \tan t$; (*d*) $s_0 \sqrt{1 - t^2}$; (*e*) $s_0 + (a/\alpha)e^\beta(e^{\alpha t} - 1)$.

(3) (*a*) $s = s_0 + v_0t + \frac{1}{2}gt^2$; (*b*) $s = a \sin t$; (*c*) $s = \frac{1}{2}a(e^t - e^{-t})$.

Art. 12.

(1) 0.73.

(2) 32.185.

(3) 0.11 ft./sec.^{2}.

(4) $j/g = 1/293$.

(5) (*a*) 0; (*b*) $2a$; (*c*) $-a/4t^{\frac{3}{2}}$; (*d*) $-ak^2 \cos kt = -k^2s$; (*e*) $ae^{-t} = s$; (*f*) $\frac{1}{2}a(e^t + e^{-t}) = s$; (*g*) $a(2t + 1)$; (*h*) $4a(3t^2 - 1)$; (*i*) $2a(3t - 1)$; (*j*) $20a(t^3 - 2)$; (*k*) $2at(t^2 + 3)/(1 - t^2)^3$.

(6) (*a*) g; (*b*) $2at$; (*c*) $2a \sin t/\cos^3 t$; (*d*) $-s_0(1 - t^2)^{-\frac{3}{2}}$; (*e*) $a\alpha e^{\alpha t+\beta}$.

Art. 19.

(1) (*a*) 128.8; (*b*) 257.6; (*e*) 144.9.

(2) 0.0917.

(4) $h = c\left[t + \frac{c}{g} - \sqrt{\frac{c}{g}\left(2t + \frac{c}{g}\right)}\right]$. An approximate value is $h = \frac{1}{2}gct^2/(c + gt)$. For a direct numerical computation the *method of successive approximations* can be used: neglecting t_2 find h approximately from $h = \frac{1}{2}gt^2$, with $t = 4$;

with this value of h find t_2, hence the time t_1, with which correct h; etc. Result: $h = 70.4$ m.

(5) (*a*) $4\frac{5}{8}$ min.; (*b*) 0.18; (*c*) $49\frac{1}{3}$; (*d*) 4 min. $4\frac{1}{2}$ sec.

(6) (*a*) v_0/g; (*b*) $\frac{1}{2}v_0^2/g$; (*c*) $2v_0/g$; (*d*) $-v_0$.

(7) (*a*) $4\frac{1}{4}$ M.; (*b*) 645 ft./sec.; (*c*) $1\frac{1}{4}$ min.; (*d*) 1200 ft./sec.; (*e*) 17, 58 sec.

(8) 80 ft./sec.

(9) (*a*) h/v_0; (*b*) $h - \frac{1}{2}gh^2/v_0^2$; (*c*) $\sqrt{gh}$.

(10) 338,000 ft./sec.; $\frac{1}{260}$ sc.

(11) 426 ft.

(12) 30 M./h.

(13) $\frac{3}{5}$ ft.

Art. 24.

(1) 5 M./sec.; $34\frac{4}{5}$ min.

(3) (*a*) 7 M./sec.; (b) 7 M./sec., 4 days $20\frac{1}{2}$ hours.

(4) $$v = -\sqrt{2gR}\sqrt{\frac{R}{s} \mp \frac{1}{k^2}}, \text{ where } \frac{v_0^2}{2gR} - \frac{R}{s_0} = \mp \frac{1}{k^2}.$$

(*a*) If $v_0 < \sqrt{2gR}\sqrt{\frac{R}{s_0}}$, $$t = \frac{k}{\sqrt{2gR}}\left[\sqrt{s(k^2R - s)} - \sqrt{s_0(k^2R - s_0)} + k^2R\left(\cos^{-1}\frac{1}{k}\sqrt{\frac{s}{R}} - \cos^{-1}\frac{1}{k}\sqrt{\frac{s_0}{R}}\right)\right];$$

(*b*) if $v_0 = \sqrt{2gR}\sqrt{\frac{R}{s_0}}$, $t = \frac{1}{3}\sqrt{\frac{2}{g}}\frac{1}{R}(s_0^{\frac{3}{2}} - s^{\frac{3}{2}})$;

(*c*) if $v_0 > \sqrt{2gR}\sqrt{\frac{R}{s_0}}$, $$t = \frac{k}{\sqrt{2gR}}\left[k^2R \log\frac{\sqrt{s + k^2R} + \sqrt{s}}{\sqrt{s_0 + k^2R} + \sqrt{s_0}} + \sqrt{s_0(s_0 + k^2R)} - \sqrt{s(s + k^2R)}\right].$$

(5) If $v_0 < \sqrt{2gR}$ the height above the earth's surface to which the particle rises is $h = v_0^2R/(2gR - v_0^2)$ and the time of rising to this height is

$$\frac{R}{2gR - v_0^2}\left(v_0 + \frac{2gR}{\sqrt{2gR - v_0^2}}\sin^{-1}\frac{v_0}{\sqrt{2gR}}\right);$$

if $v_0 = \sqrt{2gR}$, the time of rising to the distance s from the center is

$$\frac{2}{3R\sqrt{2g}}(s^{\frac{3}{2}} - R^{\frac{3}{2}}),$$

and the particle does not return; if $v_0 > \sqrt{2gR}$ the time is

$$\frac{k}{\sqrt{2gR}}\left[\sqrt{s(s+k^2)} - \sqrt{R(R+k^2)} + k^2 \log\frac{\sqrt{R} + \sqrt{R+k^2}}{\sqrt{s} + \sqrt{s+k^2}}\right],$$

where $k^2 = \dfrac{2gR^2}{v_0^2 - 2gR}$.

(6) $h = R,\ t = (1 + \frac{1}{2}\pi)\sqrt{R/g} = 34\frac{4}{5}$ min.

Art. 28.

(2) $v = \sqrt{gR} = 5$ M./sec., $T = 1$ h. 25 min.
(3) $\sqrt{s_0^2 + (v_0/\mu)^2}$.

Art. 36.

(1) π; 15.7 ft./sec.
(2) 0.157 rad./sec.^{2}; 5 rev.
(3) (a) 402; (b) 25.1 sec.
(4) (a) 0.022 rad./sec.; (b) 15.7 ft./sec.; 7.85 ft./sec.

Art. 42.

(1)–(5) Check graphically.
(6) 20″.
(7) 36 M./h.; 198 ft.
(8) $v_r = v_b \sin\theta$.
(9) Spiral of Archimedes $r = (v_0/\omega)\theta$.

Art. 48.

(4) The projection of the velocity on the radius vector and on the focal axis are in the constant ratio e of the focal radius to the distance to the directrix. It follows that the tangent meets the directrix at the same point as does the perpendicular to the radius vector through the focus.

Art. 56.

(7) $j^2 = a^2[2(1 - \cos\theta)\,\ddot{\theta} + 2\sin\theta\,\ddot{\theta}\dot{\theta}^2 + \dot{\theta}^4]$.
(8) (a) Straight line; (b) circle; (c) circle of radius v; (d) it is normal.
(9) The cylindrical components are

$$j_1 = \ddot{r}' - r'\dot{\varphi}^2, \quad j_2 = 2\dot{r}'\dot{\varphi} + r'\ddot{\varphi}, \quad j_3 = \ddot{x},$$

where $r' = r\cos\theta$. The spherical components are found by projection:

$$j_r = j_1\sin\theta + j_3\cos\theta,\quad j_\theta = j_1\cos\theta - j_3\sin\theta,\quad j_\phi = j_2.$$

Art. 59.

(3) 45°.

(5) Construct a vertical circle having the given point as its highest point and touching (*a*) the straight line, (*b*) the circle.

Art. 61.

(9) (*a*) $137\frac{1}{2}$ ft. from the vertical of the starting point; (*b*) $6\frac{1}{4}$ sec.; (*c*) 201 ft./sec., at $6\frac{1}{3}°$ to the vertical.

(10) 227.5 ft./sec. (11) 4° 22′ or 86° 48′.

(13) Let $OV = v_0$ be the given initial velocity. On the vertical through O lay off $OD = H = v_0^2/2g$; then the horizontal through D is the directrix. Make $\measuredangle\, VOF = \measuredangle\, DOV$, and lay off $OF = OD = H$; then F is the focus.

(14) With $v_0^2/2g = H$, the locus is $x^2 = -4H(y - H)$, a parabola. (17) A horizontal line.

(18) (*a*) 1.5 sec.; (*b*) 25.1 ft. from the building; (*c*) 59.7 ft./sec., at $16\frac{1}{2}°$ to the vertical.

(19) 300 ft. from tee, in 1 sec.

(20) At a distance of 6250 ft.

Art. 68.

(1) 0.99672; 86117. (4) 28.8 ft.

(2) 3.26 ft. (5) 980.4.

(3) 32.158.

(8) The pendulum should be lengthened by $\frac{1}{144}$ of its length.

(9) It will lose 67 sec./day. (10) About a mile.

Art. 70.

(3) 1.0038.

(5) Use the first formula of Art. 69.

(6) Determining the constant from $\theta = \pi$ for $v = 0$ we have $\frac{1}{2}v^2 = 2gl\cos^2\frac{1}{2}\theta$. Putting $v = -l\,d\theta/dt$ and integrating gives $t = \sqrt{l/g}\,\log\tan\frac{1}{4}(\pi + \theta)$ if $\theta = 0$ for $t = 0$.

Hence the bob approaches the highest point of the circle asymptotically, *i. e.* without reaching it in any finite time.

Art. 75.

(1) $x = x_0 \cos\mu t + (v_0/\mu) \sin\mu t$.
(2) $v = -\mu\sqrt{a^2 - x^2}$.

Art. 80.

(1) $x = 10.806 \cos(\frac{1}{6}\pi t + 27\frac{1}{3}°)$.
(2) $x = 2a \cos\frac{1}{2}\delta \cos(\omega t + \frac{1}{2}\delta)$.
(3) (*a*) $x = 2a \cos\omega t$; (*b*) $x = 0$, the case known in physics as *interference*.
(4) $x_1 = -5.18 \cos\omega t$, $x_2 = 14.14 \cos(\omega t + 30°)$.

Art. 113.

(1) $x^2 + y^2 = a^2$ being the circle, $j = -a^2 v_1^2/y^3$ where v_1 is the x-component of the initial velocity.
(2) v_0^2/a.
(3) Let $j = \mu^2 r$; then, if (x_0, y_0) is the initial position, v_1, v_2 the components of the initial velocity, the path is the hyperbola:

$$(v_2^2 - \mu^2 y_0^2)x^2 + 2(\mu^2 x_0 y_0 - v_1 v_2) + (v_1^2 - \mu^2 x_0^2)y^2 = (v_2 x_0 - v_1 y_0)^2.$$

(5) $$a = \frac{\mu}{\epsilon^2}, \quad b = \frac{v_0 r_0 \sin\psi_0}{\epsilon}, \quad \tan\alpha = -\frac{v_0^2 \sin 2\psi_0}{\epsilon^2 + v_0^2 \cos 2\psi_0},$$

where $\epsilon^2 = \dfrac{2\mu}{r_0} - v_0^2$.

(6) Put $r = 1/u$ and determine $d^2u/d\theta^2$ in terms of u alone:

$$\frac{d^2u}{d\theta^2} = -u + (n-1)(1-e^2)q^{-2n}u^{-2n+1} - (n-2)q^{-n}u^{-n+1}.$$

Hence by (16), Art. 106:

$$f(r) = c^2[(n-1)(1-e^2)q^{-2n}u^{-2n+3} - (n-2)q^{-n}u^{-n+3}].$$

$n = 1$ gives an ellipse if $e < 1$, a parabola if $e = 1$, a hyperbola if $e > 1$, all referred to focus and focal axis; $n = 2$ gives conics referred to their axes; $n = -1$ gives pascalian lima-

çons (cardioids for $e = \pm 1$); $n = -2$ gives a lemniscate if $e = \pm 1$.

(7) (*a*) $c^2(2a^2r^{-5} + r^{-3})$; (*b*) c^2r^{-3}; (*c*) $c^2(1 + n^2)r^{-3}$; (*d*) $c^2[2n^2a^2r^{-5} + (1 - n^2)r^{-3}]$.

(8) $8a^2c^2r^{-5}$.

(9) Ellipse, parabola, or hyperbola according as $\mu \gtreqless v_2{}^2y_0{}^2$, y_0 being the initial distance from the plane, v_2 the component of the initial velocity normal to the plane.

(10) $f(r) = -\dfrac{b^4}{a^2}\dfrac{v_1{}^2}{y^3}$.

Art. 137.

(3) The direction of motion passes through the highest point of the wheel.

(4) With the center O of the given circle as origin and the perpendicular to l through O as axis of x, the fixed centrode is $y^2 = cx \pm a\sqrt{x^2 + y^2}$ where a is the radius of the given circle, c the distance of O from l. With A as origin and l_1 as axis of x the body centrode is $x^2 = ay \pm c\sqrt{x^2 + y^2}$. The upper sign corresponds to l_1 sliding over the first and second quadrants of the circle, the lower to l_1 sliding over the third and fourth quadrants. If $c > a$, the complete fixed centrode has a node at O with the tangents $ay = \pm\sqrt{c^2 - a^2}x$. The polar equations of the centrodes are $r\sin^2\theta = c\cos\theta + a$ and $r'\cos^2\theta' = a\sin\theta' + c$. The body centrode for $c > a$ is (apart from position) the same curve as the fixed centrode for $a > c$, and vice versa.

(5) $y^2 = 2a(x + \frac{1}{2}a)$.

(6) The fixed centrode is a circle passing through O_1, O_2; the body centrode is a circle of twice the radius of the fixed centrode. The path of any point in the fixed plane is a Pascal limaçon; the points of the body centrode describe cardioids.

(8) Two equal parabolas; the motion is the same as that of Ex. (5).

(10) With O as pole and OB as polar axis the equation of the fixed centrode is $r^2\cos^2\theta - 2ar\cos^2\theta + a^2 = l^2$. With O as origin and OB as axis of x the equation is $(x^2 + a^2$

$-\ l^2)\sqrt{x^2+y^2} = 2ax^2$. The rationalized equation represents the centrode of AB when B moves not only on the positive but also on the negative half of the axis of x. The equation of the body centrode, with A as pole and AB as polar axis, $AC = r'$, $\measuredangle BAC = \theta'$, is found by observing that $r = r' + a$, $l \sin\theta' = OB \sin\theta = r \cos\theta \sin\theta$ whence

$$(a^2 - l^2 \cos^2\theta')r'^2 - 2al^2r' \sin^2\theta' + l^2(l^2 - a^2 \cos\theta') = 0,$$

i. e.

$$r_1' = l\frac{l + a\cos\theta'}{a + l\cos\theta'}, \quad r_2' = l\frac{l - a\cos\theta'}{a - l\cos\theta'}.$$

These relations can be read off directly from the figure if perpendiculars be dropped from O on AB and from B on AC. For the path of any point P whose body co-ordinates, with A as origin and AB as axis of x', are x', y', we have

$$x = a\cos\theta + x'\cos\varphi + y'\sin\varphi, \quad y = a\sin\theta - x'\sin\varphi + y'\cos\varphi,$$

where θ and φ are connected by the relation $l/a = \sin\theta/\sin\varphi$. For the path of the midpoint of AB we find $x = a\cos\theta + \frac{1}{2}l\cos\varphi$, $y = a\sin\theta - \frac{1}{2}l\sin\varphi$, whence $x = \sqrt{a^2 - 4y^2} + \frac{1}{2}\sqrt{l^2 - 4y^2}$ which is of the fourth degree.

To find the velocity of B when that of A is given observe that as the distance AB is invariable the projections of the velocities of A and B on AB must be equal, whence $v_B \cos\varphi = v_A \sin(\theta + \varphi)$.

(12) Find first the velocity v_r of P_2 relative to P_1 as the resultant of $-v_1$ and v_2; hence ω.

Art. 148.

(2) A Pascal limaçon.

(7) (*a*) $\omega^2 x - \dot{\omega} y = 0$; (*b*) $\dot{\omega} x - \omega^2 y + \bar{j} = 0$.

Art. 166.

(2) Distance from midpoint $= \frac{1}{3}\frac{8}{5}l$, $2l$ being the distance of 5 from 23.

(3) About 1000 M. below the earth's surface.

(5) $\bar{x} = r \sin\alpha/\alpha = rc/s$, where c is the chord, s the arc; for the semi-circle $\bar{x} = (2/\pi)r$.

(6) $\bar{x} = \frac{1}{4}\dfrac{3\sqrt{2} - \log(1 + \sqrt{2})}{\sqrt{2} + \log(1 + \sqrt{2})}\, a = 0.40496a,$

$$\bar{y} = \frac{4}{3}\,\frac{2\sqrt{2} - 1}{\sqrt{2} + \log(1 + \sqrt{2})}\, a = 1.12907a.$$

(7) $\pi a,\ \frac{4}{3}a.$ (8) $\frac{4}{5}a,\ \frac{4}{5}a.$
(9) $r\sin\theta/\theta,\ r(1 - \cos\theta)/\theta,\ \frac{1}{2}kr\sin\theta.$
(10) $2a(\alpha\sin\alpha + \cos\alpha - 1)/\alpha^2,\ 2a(\sin\alpha - \alpha\cos\alpha)/\alpha^2.$
(12) $\frac{61}{116}a,\ \frac{63}{116}a.$
(13) Distance from hypotenuse $= 0.11a.$
(15) (*a*) $\frac{3}{5}x_1,\ \frac{3}{8}y_1$; (*b*) $\frac{1}{2}\pi,\ \frac{1}{8}\pi$; (*c*) $\dfrac{4}{3\pi}a = 0.40531a,\ \dfrac{4}{3\pi}b$;
(*d*) $\dfrac{2a}{3(\pi - 2)} = 0.564a,\ \dfrac{2b}{3(\pi - 2)}.$
(17) $\frac{2}{3}r\sin\alpha/\alpha.$ (18) $\frac{1}{5}a.$
(22) If x_1, x_2 are the distances of the planes from the center then $\bar{x} = \frac{1}{2}(x_1 + x_2)\dfrac{a^2 - \frac{1}{2}(x_1^2 + x_2^2)}{a^2 - \frac{1}{3}(x_1^2 + x_1x_2 + x_2^2)}.$
(*a*) $\frac{3}{4}(2a - h)^2/(3a - h)$; (*b*) $\frac{3}{8}a$; $\frac{3}{8}a(1 + \cos\alpha).$
(23) $\frac{2}{3}h.$ (25) $\frac{5}{16}y_1.$
(24) $\frac{5}{12}y_1.$ (26) $\frac{3}{8}a,\ \frac{3}{8}b,\ \frac{3}{8}c.$
(27) (*a*) $\frac{4}{3}a,\ \frac{4}{3}a$; (*b*) $\dfrac{16 + 9\pi^2}{18\pi}a = 1.85374a,\ \frac{5}{6}a$; (*c*) $\frac{26}{15}a$;
(*d*) $\dfrac{128 + 454}{90\pi}a = 2.0412a.$ (28) $\dfrac{n + 3}{2(n + 4)}a.$

Art. 170.

(1) 300,000 F.P.S. units. (3) 32,000 F.P.S. units.
(2) 50 ft./sec.

Art. 179.

(1) 6.4×10^5 poundals $= 8.9 \times 10^9$ dynes.
(2) 4.5 pounds. (3) 0.1406.

Art. 196.

(1) $\theta = 120°.$ (3) 28, 39° 16′.
(4) $2F\cos 22\frac{1}{2}° = 1.848F.$

(7) (a) $W \sin\theta$, $W \cos\theta$; (b) $W \tan\theta$, $W \sec\theta$; (c) $W \sin\theta \sec\alpha$, $W \cos(\theta + \alpha) \sec\alpha$.

(8) $W \sin\beta/\sin(\alpha + \beta)$, $W \sin\alpha/\sin(\alpha + \beta)$.

(10) $P = \frac{4}{5}W$, $T = \frac{5}{3}W$.

(11) $P = 2W \cos\frac{1}{2}(\alpha + \frac{1}{2}\pi) = 0.8945W$.

(12) $P = W \sin(\alpha + \beta)\sin\beta$ is greatest when the sail bisects the angle between the wind and the direction of motion.

(13) $W \sin\beta/\sin(\alpha + \beta)$, $W \sin\alpha/\sin(\alpha + \beta)$.

(14) $T = Wl/d$, $T' = W(c - l)/d$, where $d^2 = l^2 - \frac{1}{4}(c - l)^2$.

(15) 13.4, 28.9, 50, 86.6, 186.6, ∞.

(16) 848, 282; 1000, 600. (17) $0.640W$.

(18) (a) $1.414W$; (b) $2W \cos\frac{1}{2}(\frac{1}{2}\pi \pm \theta)$.

Art. 221.

(2) $T = mW$, $A = \sqrt{m^2 - m + 1}\ W$, where $m = 2c/l$.

(3) $F = \frac{1}{2}(\cot\theta - r/l)W$.

(4) $\tan\theta = (a \cot\alpha - b \cot\beta)/(a + b)$.

(5) $$A_x = -B = -\frac{ac}{l\sqrt{l^2 - a^2}}W, \quad A_y = W.$$

(6) $$A_x = -\frac{ac\sqrt{l^2 - a^2}}{l^3}W, \quad A_y = \left(1 - \frac{a^2c}{l^3}\right)W, \quad B = \frac{ac}{l^2}W.$$

Art. 243.

(1) $P = W \sin\varphi/\cos(\alpha - \varphi)$.

(2) (a) $\dfrac{\sin(\theta - \varphi)}{\cos\varphi} < \dfrac{P}{W} < \dfrac{\sin(\theta + \varphi)}{\cos\varphi}$; (b) $0 \leqq \dfrac{P}{W} \leqq 2 \sin\theta$; (c) if P act up the plane, $P \lesseqgtr \dfrac{\sin(\theta + \varphi)}{\cos\varphi} W$; if P act down the plane, $P \lesseqgtr \dfrac{\sin(\varphi - \theta)}{\cos\varphi} W$.

(4) $226\frac{1}{4}$, $56\frac{1}{2}$. (5) $\theta = \frac{1}{2}\pi - 2\varphi$.

(6) $\mu = l \sin\theta \cos\theta/(c - l \cos^2\theta)$.

(7) $A = mW \sin(\theta - \varphi) \cos\theta/\sin\varphi$, $C = mW \cos\theta$, $\tan 2\varphi = m \sin 2\theta$, where $m = l/c$.

(8) $\sin\theta \leqq \frac{3}{8}$.

Art. 247.

(1) 33×10^{-12}.

Art. 254.

(8) $\dfrac{2\kappa\rho''^2}{c}[\sqrt{c^2 + (a+b)^2} - \sqrt{c^2 + (a-b)^2}]$.

(10) $2\pi\kappa\rho'\left[\dfrac{p+c}{\sqrt{a^2+(p+c)^2}} - \dfrac{p}{\sqrt{a^2+p^2}}\right]$; in the limit: $2\pi\kappa\rho' c\,\dfrac{a^2}{(a^2+p^2)^{\frac{3}{2}}}$.

Art. 260.

(3) $2\pi\kappa\rho'(\sqrt{x^2+a^2} - x)$.

(5) At the distance x from the center, if c is the radius of the circle,

$$U = \frac{4\kappa\rho'' c}{x+c}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},\ \text{where } \kappa = \frac{\sqrt{cx}}{\frac{1}{2}(c+x)}.$$

(6) $U = cz + C$. (7) $U = mg(z_0 - z)$.

(9) $U = -\int f(r)dr$; equipotential surfaces $r = c$.

Art. 264.

(3) $7500,\ 101.7 \times 10^9$. (5) 150 ft.-lb.

(4) 18,000 ft.-lb.

Art. 290.

(1) (*a*) $\frac{1}{2}$ lb.; (*b*) 11.3 ft./sec.; (*c*) 0.63 sec.; (*d*) $\frac{1}{8}$ ft.-lb.

(4) If $x_0 < e$ nothing is changed; if $x_0 > e$ the particle performs simple harmonic oscillations about Q_1.

(5) The length l is increased to $l + e + \sqrt{e(e+2h)}$.

(7) 42 min. 35 sec.

Art. 293.

(2) The equation of motion $\ddot{s} = \dot{v} = -g - kv^2$ gives with $k = \mu^2/g$:

$$v = \frac{g}{\mu}\,\frac{\mu v_0 \cos\mu t - g \sin\mu t}{\mu v_0 \sin\mu t + g \cos\mu t},$$

$$s = \frac{g}{\mu^2}\log\left(\frac{\mu v_0}{g}\sin\mu t + \cos\mu t\right) = \frac{1}{2k}\log\frac{g + kv_0^2}{g + kv^2}.$$

(3) $t = \dfrac{1}{\sqrt{kg}}\tan^{-1}\left(\sqrt{\dfrac{k}{g}}\,v_0\right), \quad h = \dfrac{1}{2k}\log\left(1 + \dfrac{k}{g}v_0^2\right).$

(4) $\dfrac{v_1}{v_0} = \dfrac{\sqrt{g}}{\sqrt{g + kv_0^2}}.$

(5) In vacuo $v = 139$ ft./sec., in air $v = 122$ ft./sec.

(6) $s = \dfrac{v_0}{k}(1 - e^{-kt}), \quad v = v_0 e^{-kt} = v_0 - ks.$

(7) $v = \dfrac{g}{k}(1 - e^{-kt}),$

$$s = \frac{g}{k}\left[t + \frac{1}{k}(e^{-kt} - 1)\right] = -\frac{v}{k} + \frac{g}{k^2}\log\frac{g}{g - kv}.$$

Art. 297.

(2) The logarithmic decrement is $\log e^{-\lambda t} = -\lambda t$.

(4) If $\mu \neq \kappa, \quad s = c_1 \cos\kappa t + c_2 \sin\kappa t + \dfrac{a}{\kappa^2 - \mu^2}\sin\mu t$; if $\mu = \kappa, \quad s = c_1 \cos\kappa t + c_2 \sin\kappa t + \dfrac{at}{2k}\sin\kappa t.$

(5) The term due to the forced oscillation is

$$\frac{a}{\sqrt{(\kappa^2 - \mu^2)^2 + 4\lambda^2\mu^2}}\cos\mu(t - t_0);$$

hence the oscillation lags behind the force by the phase difference μt_0; the amplitude is less than for undamped oscillations. The free oscillations (if any) will rapidly die out so that the motion soon approaches the state of motion given by the above term.

Art. 302.

(2) The equation of the orbit given in Ex. (1) is satisfied not only by x_0, y_0, but also by $v_1/\kappa, v_2/\kappa$; *i. e.* the orbit passes not only through the initial position P_0, but also through the point $Q(v_1/\kappa, v_2/\kappa)$ which is the extremity of the radius

vector $OQ = v_0/\kappa$ parallel to v_0; OP_0 and OQ are the conjugate semi-diameters whose equations are $x_0y = y_0x$, $v_1y = v_2x$.

(4) The problem requires the construction of the axes of a conic from a pair of conjugate diameters.

(5) Referring the orbit to its axes we have $x = a \cos\kappa t$, $y = b \sin\kappa t$ for the ellipse and $x = \frac{1}{2}a(e^{\kappa t} + e^{-\kappa t}) = a \cosh\kappa t$, $y = \frac{1}{2}b(e^{\kappa t} - e^{-\kappa t}) = b \sinh\kappa t$ for the hyperbola.

(6) From the equations of Ex. (5) it follows that for the ellipse $\tan\theta = (b/a) \tan\kappa t$ whence $\dot{\theta} = \kappa ab/r^2$.

(8) Use the equations of the conic in terms of the eccentric angle φ.

(9) (*a*) Ellipse; (*b*) hyperbola; (*c*) parabola.

(10) The parabola $x - x_0 = (v_1/v_2)(y - y_0) - (2\kappa c/v_2^2)(y - y_0)^2$, where $2c$ is the distance of O_3 from the point O that bisects O_1O_2; the midpoint between O and O_3 is taken as origin and OO_3 as axis of x.

(11) $t = \dfrac{1}{\kappa}\tan^{-1}\left(\dfrac{a}{b}\tan\theta\right)$.

Art. 320.

(1) $v_0 = \sqrt{\mu/r_0}$. (3) 687 days.

(4) As the velocity is not changed instantaneously we have by (24), Art. 314:

$$\frac{2\mu}{r} \mp \frac{\mu}{a} = \frac{2\mu'}{r} \mp \frac{\mu'}{a'},$$

whence a' is found.

(5) An ellipse, with the end of its minor axis at the point where the change takes place.

(6) (*a*) Ellipse with $a = \frac{2}{3}r$; (*b*) parabola.

(7) Differentiate (24), Art. 314, with respect to μ and a.

(8) The periodic time T would be diminished by $(2/n)T$.

(9) $r = l/(1 + e \cos\theta)$ gives x, y as functions of θ; hence, observing that $r^2\dot{\theta} = c$, $\dot{x} = -(c/l)\sin\theta$, $\dot{y} = (c/l)(\cos\theta + e)$. The hodograph is therefore the circle $\dot{x}^2 + (\dot{y} - ce/l)^2 = (c/l)^2$, where $c = \sqrt{\mu l}$.

(10) 1.034 114.

(11) $t = \sqrt{2a^3/\mu}(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta)$.

(12) 178.73 and 186.52 days.

Art. 334.

(1) (*a*) $7\frac{1}{2}$ lb.; (*b*) 480 lb.; (*c*) 6.4 rev./sec.
(2) $8\frac{1}{2}°$.
(4) 32.20.
(7) $\tan\delta = R\omega^2\sin\varphi\cos\varphi/(g - R\omega^2\cos^2\varphi)$; 44°57′.
(8) $7\frac{1}{2}$ lb.

Art. 339.

(4) To count the angles from the highest point of the circle put $\pi - \theta = \varphi$; then, putting $h - l = h'$, where h' is the height to which the velocity at the highest point is due, we have $N = -3mg[\cos\varphi - \frac{2}{3}(1 + h'/l)]$. The particle remains on the curve as long as $\cos\varphi > \frac{2}{3}(1 + h'/l)$; distinguish the cases $h' \gtreqless 0$.

(6) At the distance $1.4617a$ from the lowest point of the circle if a is the radius.

Art. 379.

(1) $\frac{1}{3}l^2$.
(2) (*a*) $\frac{1}{3}l^2$; (*b*) $\frac{1}{3}h^2$; (*c*) $\frac{1}{12}l^2$; (*d*) $\frac{1}{12}h^2$.
(3) $\frac{1}{2}h^2$.
(4) $\frac{1}{12}a^2$.
(5) $\frac{5}{24}a^2$.
(6) $\frac{1}{12}h^2$.
(7) $\dfrac{n+1}{n+3}l^2$.
(8) $\frac{1}{4}a^2$.
(9) $\frac{1}{2}a^2$.
(10) $\frac{1}{5}a^2$.
(11) $\frac{1}{5}a^2$, $\frac{1}{5}b^2$, $\frac{1}{5}c^2$.
(12) $\frac{1}{4}(a_1{}^2 + a_2{}^2)$

Art. 386.

(1) $\frac{1}{12}(h^2 + l^2)$.
(2) (*a*) $\frac{1}{24}a^2$; (*b*) $\frac{1}{24}a^2$; (*c*) $\frac{1}{12}a^2$.
(3) (*a*) $\frac{5}{4}a^2$; (*b*) $\frac{1}{2}a^2$; (*c*) $\frac{3}{2}a^2$.
(4) $\frac{2}{5}a^2$.
(5) $\frac{1}{2}(a_1{}^2 + a_2{}^2)$.
(6) (*a*) $\frac{1}{2}a^2$; (*b*) $\frac{3}{2}a^2$; (*c*) $\frac{1}{12}(h^2 + 3a^2)$.
(8) $\frac{2}{3}a^2$.
(9) (*a*) $\frac{1}{4}b^2$; (*b*) $\frac{1}{4}a^2$; $\frac{1}{4}(a^2 + b^2)$.
(10) $\frac{1}{5}(b^2 + c^2)$, $\frac{1}{5}(c^2 + a^2)$, $\frac{1}{5}(a^2 + b^2)$.
(11) $\frac{1}{6}a^2$.
(12) $\frac{4}{3}a^2$.
(13) $\frac{3}{4}a^2 + b^2$.

Art. 403.

(1) The centroidal principal axes are perpendicular to the faces. The moments for these axes are $\frac{1}{3}M(b^2 + c^2)$, $\frac{1}{3}M(c^2 + a^2)$, $\frac{1}{3}M(a^2 + b^2)$. The central ellipsoid is $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = 3c^4$. For an edge

$2a$, $I = \frac{4}{3}M(b^2 + c^2)$; for a diagonal $I = \frac{2}{3}M(b^2c^2 + c^2a^2 + a^2b^2)/(a^2 + b^2 + c^2)$.

For the cube the fundamental ellipsoid becomes a sphere of radius $\frac{1}{3}\sqrt{6}a$; for an edge of the cube, $q^2 = \frac{8}{3}a^2$; for a diagonal, $q^2 = \frac{2}{3}a^2$.

(2) Central ellipsoid: $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = 5\epsilon^4$; for l, $q^2 = \frac{1}{5}(6a^2 + b^2)$.

(3) Take the vertex as origin, the axis of the cone as axis of x; then $I_1 = \frac{3}{10}Ma^2$; I_1', *i. e.* the moment of inertia for the yz-plane, $= \frac{3}{5}Mh^2$. As for a solid of revolution about the axis of x $B' = C'$ and $B = C$, we have $I_2' = I_3' = \frac{1}{2}I_1$, and $I_2 = I_3 = I_1' + \frac{1}{2}I_1$. Hence, $I_2 = I_3 = \frac{3}{5}M(h^2 + \frac{1}{4}a^2)$. At the centroid the squares of the principal radii are $\frac{3}{10}a^2$, $\frac{3}{80}(4a^2 + h^2)$.

(4) $A = B = C = \frac{2}{3}Ma^2$, $D = E = F = \frac{1}{4}Ma^2$; hence momental ellipsoid: $4(x^2 + y^2 + z^2) - 3(yz + zx + xy) = 6\epsilon^4/a^2$; squares of principal radii: $\frac{1}{6}a^2$, $\frac{11}{12}a^2$, $\frac{11}{12}a^2$.

(5) $q^2 = \frac{1}{2}a^2(1 + \sin^2\alpha)$.

(6) $I = \frac{1}{10}\rho\pi a^4(\frac{8}{3}a + H + 2h^5/H^4)$; for $h = a = \frac{1}{3}H$, $q^2 = \frac{461}{1410}a^2$.

(7) $A = I_1$, $B = I_2 + Mx_1^2$, $C = I_3 + Mx_1^2$.

(8) The centroid may be such a point; if the central ellipsoid be an oblate spheroid, the two points on the axis of revolution at the distance $\pm\sqrt{(I_1 - I_2)/M}$ from the centroid are such points.

(9) The ellipsoid must have the same central ellipsoid as the given body; its equation is $x^2/A' + y^2/B' + z^2/C' = 5/M$, where M is the mass and A', B', C' are the moments of inertia for the principal *planes* of the body at the centroid.

(10) $\rho'' = M/N$, where

$$N = \sqrt{6}[(q_2^2+q_3^2-q_1^2)^{\frac{1}{3}}+(q_3^2+q_1^2-q_2^2)^{\frac{1}{3}}+(q_1^2+q_2^2-q_3^2)^{\frac{1}{3}}]^{\frac{3}{2}}.$$

$$a^3 = \frac{3}{4}\frac{M}{\rho''}(q_2^2 + q_3^2 - q_1^2), \text{ etc.}$$

Art. 420.

(1) $2\pi\sqrt{\dfrac{4L^2 + 3a^2}{6gL}}$.

(2) $\frac{2}{3}\sqrt{2}a$.

(3) $m(\frac{5}{3}\pi r)^2$.

(4) $\sqrt{3g/l}$

(5) $\frac{4}{7}a$.

(6) $\frac{1}{12}$ lb.

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